CS401 - Problem Set 2

1. If we discussed revisions to your learning plan, please make those changes.

2. In this problem, you'll explore how the base used in writing down the language affects the complexity class the language is in.

   (a) For $S \subseteq \mathbb{N}$, let $L_b^S = \{n_b : n \in S\}$, where $n_b \in \{0, 1, \ldots, b-1\}^*$ is a string that represents the natural number n written in base b. Prove that for $b > 2$, if $L_b^S \in \mathbf{P}$, then $L_2^S \in \mathbf{P}$.

   (b) Explain (using as plain, accessible language as possible) why your proof in part b justifies our choice to only consider binary languages in this class (at least in cases where we have at least polynomial time as a resource).

   (c) * Prove that the following language is in $\mathbf{P}$:

   $$\mathsf{UNARYFACTORING} = \{\langle n_{\text{unary}}, l_{\text{unary}}, k_{\text{unary}}\rangle : \exists j \in \mathbb{N} \text{ s.t. } l \leq j \leq k \text{ and } j \text{ divides } n\}. \tag{1}$$

   In this case, n_{unary} means the number n represented in unary. A unary string is an element of $\{1\}^*$ where a natural number is represented using a string containing that number of ones. So for example, 2 is represented in unary as 11, and 5 is represented in unary as 11111. Then $\langle 1111111111, 1111, 1111111\rangle \in \mathsf{UNARYFACTORING}$ because in base 10, this corresponds to the sequence $< 10, 4, 7 >$. Since 5 divides 10 and $4 \leq 5 \leq 7$, this tuple satisfies the conditions, and so is in the language. On the other hand, $\langle 111111111, 1111, 1111111\rangle \notin \mathsf{UNARYFACTORING}$ because there is no number between 4 and 7 that divides 9.

   (d) It is unknown whether the problem of factoring in base 2 is in $\mathbf{P}$. However, in part (c) you prove that $\mathsf{UNARYFACTORING} \in \mathbf{P}$. Why can't we use the same argument as in part (a) to convert base-2 factoring into $\mathsf{UNARYFACTORING}$, and thus prove factoring is in $\mathbf{P}$?

3. **(Extra Practice Problem)** Let $L_\triangle = \{G : G \text{ contains a triangle}\}$. Prove $L_\triangle$ is in $\mathbf{P}$.

4. **(Moved to Pset 3)**

   (a) A unary language L is a subset of $\{1\}^*$. Let $\mathbf{NP}_U$ be the set of unary languages that are also in $\mathbf{NP}$. Prove that if $\mathbf{NP}_U \subseteq \mathbf{P}$, then $\mathbf{EXP} = \mathbf{NEXP}$, where $\mathbf{NEXP}$ is defined similarly to $\mathbf{NP}$ except now the TM can run for exponential time in the size of the input, and the witness u can be of exponential size in the size of the input.

   (b) $\mathbf{EXP}$ vs $\mathbf{NEXP}$ is the exponential time version of the $\mathbf{P}$ vs $\mathbf{NP}$ question. Based on the result in part (a), how does the $\mathbf{EXP}$ vs $\mathbf{NEXP}$ question relate to the $\mathbf{P}$ vs $\mathbf{NP}$ question? Is this relationship surprising?

5. **(You almost, but not quite, have the tools to solve this. I'll move it to next pset, but I encourage you to think about it or try to solve it on this pset.)** Prove if $L' \in \mathsf{NP}$ and $L \leq_p L'$, then $L \in \mathsf{NP}$.

6. Let $L_{ksort} = \{\langle x, k\rangle : x$ is a list of unsorted integers, and the k^{th} largest integer is odd$\}$. Prove L_{ksort} is in P.