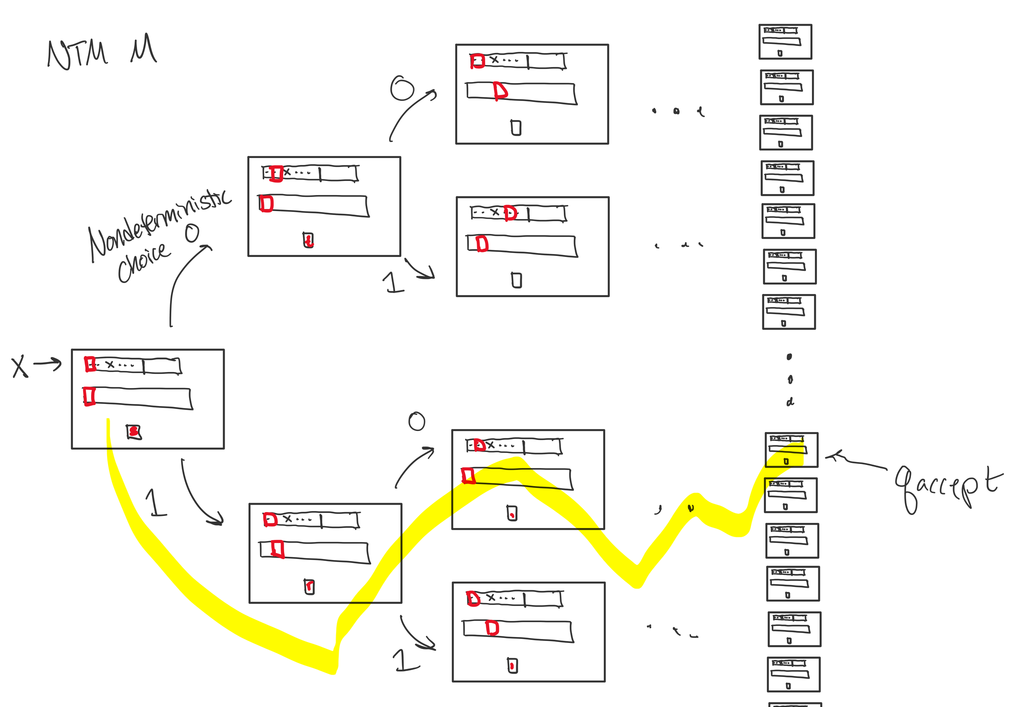
9 Probabilistic Computation

Wednesday, April 20, 2022
1:19 PM

Recall Non-deterministic Polynomial Time Computation:

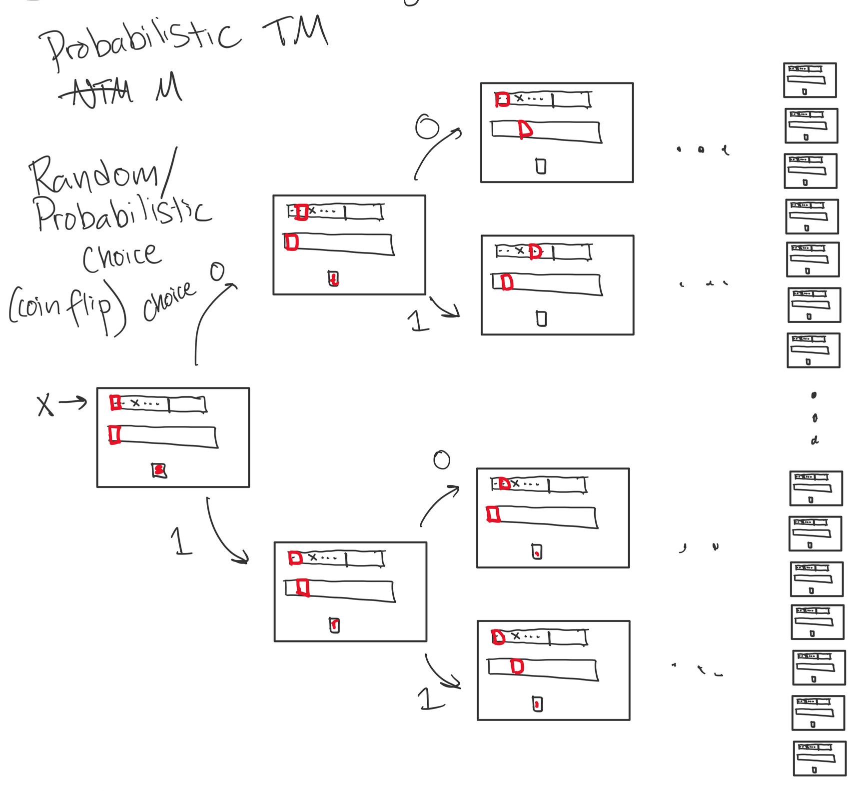


Polynomial Steps

- u is sequence of nondeterministic choices
- $M(x,u)$ is output of following u

$$\boxed{\begin{array}{l} M \text{ accepts } x \text{ if } \exists\, u:\ M(x,u) \text{ accepts} \\ M \text{ rejects } x \text{ if } \forall\, u:\ M(x,u) \text{ rejects} \end{array}}$$

Probabilistic Polynomial Time Computation:



Each time run $M(x)$, get one of these outcomes with equal probability

EXISTS!

Polynomial Steps

- u is sequence of ~~nondeterministic~~ probabilistic choices
- $M(x,u)$ is output of following u

$$\boxed{\begin{array}{l} M \text{ accepts } x \text{ if } \Pr_u[M(x,u)=\text{accept}] \geq \frac{2}{3} \\ M \text{ rejects } x \text{ if } \Pr_u[M(x,u)=\text{accept}] \leq 1/3 \end{array}}$$

⇓

At most 1/3 of paths accept

BPP (Bounded Probabilistic Polynomial Time)

$L \in BPP$ if $\exists$ a probabilistic TM M s.t. M halts in polynomial time regardless of its random choices, and $\forall x \in \{0,1\}^n$

- If $x \in L$ $\Pr[M(x,u)=\text{accept}] \geq 2/3$
- If $x \notin L$ $\Pr[M(x,u)=\text{reject}] \geq 2/3$

Why 2/3? Arbitrary. What is important?

- If $x \in L$ $\Pr[M(x,u)=\text{accept}] \geq \frac{1}{2}+\varepsilon$ ← constant
- If $x \notin L$ $\Pr[M(x,u)=\text{reject}] \geq \frac{1}{2}+\varepsilon$

"Bounded away from 1/2"

As long as bounded, can do:

- Repeat M $O(k)$ times
- Take majority vote

- If $x \in L$ $\Pr[M(x,u)=\text{accept}] \geq 1 - O(\frac{1}{2^k})$
- If $x \notin L$ $\Pr[M(x,u)=\text{reject}] \geq 1 - O(\frac{1}{2^k})$

RP (Randomized Polynomial Time)

$L \in RP$ if $\exists$ a probabilistic TM M s.t. M halts in polynomial time regardless of its random choices, and $\forall x \in \{0,1\}^n$

- If $x \in L$ $\Pr[M(x,u)=\text{accept}] \geq 2/3$
- If $x \notin L$ $\Pr[M(x,u)=\text{accept}] = 0$

- If M decides $L \in RP$, and $M(x,u)$ accepts for a sequence of random choices u, what do you know?

A) $x \in L$ B) $x \notin L$ C) Nothing

- If M decides $L \in RP$, and $M(x,u)$ rejects for a sequence of random choices u, what do you know?

A) $x \in L$ B) $x \notin L$ C) Nothing

Example

$L = \{x \in \{0,1\}^*:\ x \text{ contains at least one } 1\}$

Let M be the PTM that looks a $2n/3$ bits of x, chosen at random.

$x \in L \rightarrow \Pr(M \text{ accepts}) \geq 2/3$

$x \notin L \rightarrow \Pr(M \text{ rejects}) = 1$

Draw Map + Explain:

P, BPP, RP, NP, PSPACE