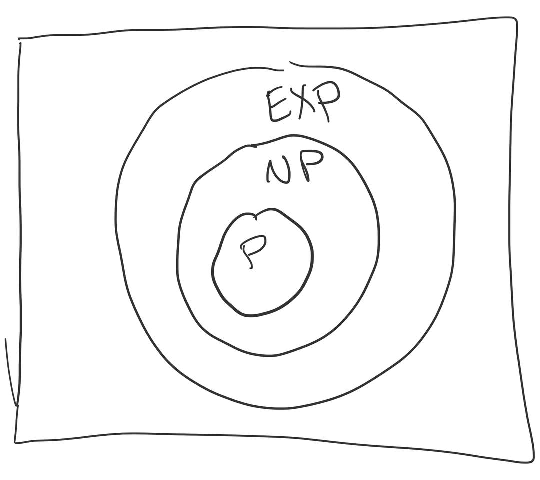
7s Separations

Thursday, April 7, 2022 2:48 PM



$P \subseteq NP$ ✓

$P = NP$?

$NP \subseteq EXP$ ✓

$NP = EXP$?

$P \neq EXP$ $\boxed{\text{No}}$

Strategy: Diagonalization

ex: undecidable languages (301)

ex: $|\mathbb{Z}| < |\mathbb{R}|$ (200)

Tool: Universal TM $\mathcal{U}$ $\Rightarrow$ $\mathcal{U}$ simulates any other TM efficiently

Facts:

① $\forall \alpha \in \{0,1\}^*$, α represents a TM M_α

② $\forall$ TM M, there are infinitely many α s.t. $M = M_\alpha$

③ $\mathcal{U}$ takes as input $\langle \alpha, x \rangle$ (M_α, input) and $\mathcal{U}(\alpha,\alpha)$

$\mathcal{U}(\alpha, x) = M_\alpha(x)$ $\mathcal{U}(M_\alpha, x)$

- $\exists$ c' (a constant) s.t.
  - If $M_\alpha(x)$ takes time t
  - then $\mathcal{U}(\alpha,x)$ takes time $c' t \log t$

Theorem: $DTIME(n) \subset DTIME(n^{1.5})$

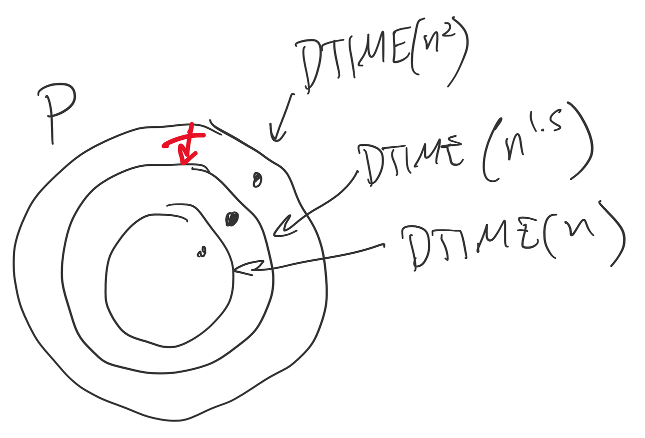
Idea: Create L s.t. $L \in DTIME(n^{1.5})$ but $L \notin DTIME(n)$.

Step 1: Creat L

input string x

TM α

f	y	z	w
y	0	1	1
z	1	1	0
w	0	0	0



$$f(\alpha, x) = \begin{cases} 0 & \text{if } \mathcal{U}(\alpha,x) \text{ does not terminate in } |x|^{1.4} \text{ steps, or rejects in } |x|^{1.4} \text{ steps} \\ 1 & \text{if } \mathcal{U}(\alpha,x) \text{ accepts in } |x|^{1.4} \text{ steps} \end{cases}$$

$L = \{x : f(x,x) = 0\}$ ex: $L = \{y, w, \ldots\}$

Step 2: $L \in DTIME(n^{1.5})$

- Run $\mathcal{U}(x,x)$ for $|x|^{1.4}$ steps
- Accept if $\mathcal{U}$ has not terminated, or if $\mathcal{U}(x,x)$ rejects
- Reject otherwise

Step 3: $L \notin DTIME(n)$

- For contradiction, assume $L \in DTIME(n)$
- Then $\exists$ a TM M s.t. $M(x)$ accepts iff $x \in L$ and $M(x)$ always terminates in $c|x|$ steps for some constant c.

$\boxed{\diamondsuit}$ • This means $\forall \alpha$ s.t. $M_\alpha = M$, $\mathcal{U}(\alpha,\alpha)$ terminates in $c' t \log t = c' c|\alpha| \log(c|\alpha|)$

- Pick α^* s.t. Pick a large enough α^* s.t. is true
  - $M_{\alpha^*} = M$

$\boxed{\heartsuit}$ • $|\alpha^*|^{1.4} > c' c|\alpha^*| \log(c|\alpha^*|)$

$\boxed{\diamondsuit} + \boxed{\heartsuit} \Rightarrow \mathcal{U}(\alpha^*, \alpha^*)$ terminates in less than $|\alpha^*|^{1.4}$ steps

Case 1: $\alpha^* \in L$

- By def of L, $f(\alpha^*, \alpha^*) = 0$
- By def of f, $\mathcal{U}(\alpha^*,\alpha^*)$ doesn't terminate in $|\alpha^*|^{1.4}$ steps, or rejects
- By $\diamondsuit + \heartsuit$, $\mathcal{U}(\alpha^*,\alpha^*)$ terminates in less than $|\alpha^*|^{1.4}$ steps, so $\mathcal{U}(\alpha^*,\alpha^*)$ must reject in at most $|\alpha^*|^{1.4}$ steps
- By def $\mathcal{U}(\alpha^*,\alpha^*)$

$$\underbrace{\mathcal{U}(\alpha^*,\alpha^*)}_{\text{rejects}} = M_{\alpha^*}(\alpha^*) = \underbrace{M(\alpha^*)}_{\text{rejects}}$$

$\Leftarrow$ a contradiction, since M decides L.

Case 2: $\alpha^* \notin L$

- By def of L, then $f(\alpha^*,\alpha^*) = 1$.
- By def of f, $\mathcal{U}(\alpha^*,\alpha^*)$ accepts in $|\alpha^*|^{1.4}$ steps.
- By def $\mathcal{U}(\alpha^*,\alpha^*)$

$$\underbrace{\mathcal{U}(\alpha^*,\alpha^*)}_{\text{accepts}} = M_{\alpha^*}(\alpha^*) = \underbrace{M(\alpha^*)}_{\text{accepts}}$$

, a contradiction, since M decides L.