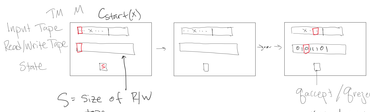
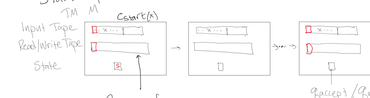
**Theorem:** TQBF is PSPACE-hard

$\forall L \in PSPACE,\ L \le_p TQBF$

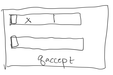
**Pf:** Let $L \in PSPACE$. Then $\exists$ a TM $M$ st




Then there also exists a TM $M'$ that decides $L$ in $C|x|^d$ but erases R/W tape at end, heads at start.




$C_{accept}(x)$ = unique accepting config:

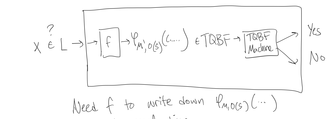



**def:** $\varphi_{M',i}(C,C')$ = True if there is a path from configuration $C$ to $C'$ in $M'$ of length at most $2^i$

What is the longest possible path from $C_{start}$ to $C_{accept}$? A) $O(\log S)$ B) $O(S)$ C) $O(2^S)$ D) $2^{O(S)}$

# of configurations

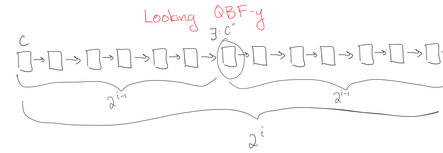
Strategy: $x \in L$ iff $\varphi_{M',O(S)}(C_{start}(x), C_{accept}(x)) = T$ (QBF)




Need $f$ to write down $\varphi_{M,O(S)}(\dots)$ in polynomial time

Claim: $\varphi_{M',i}(C,C') = \exists$ config $C'': \varphi_{M',i-1}(C,C'') \wedge \varphi_{M',i-1}(C'',C')$

Looking QBF-y




Idea: $\varphi_{M',i}(C_{start}(x), C_{accept}(x)) =$

$\exists C'': \varphi_{M,i-1}(C_{start}(x), C'') \wedge \varphi_{M,i-1}(C'', C_{accept})$

$\exists C'', C': \varphi_{M',i-2}(C_{start}(x), C') \wedge \varphi_{M',i-2}(C', C'')$ ...

$\vdots$

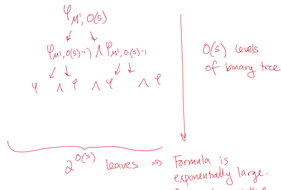
$\exists C_1, C_2, \dots : \varphi_{M',0}(C_i, C_j) \wedge \varphi_{M',0}(C_j, C_k) \cdots$

Fact: $\varphi_{M',0}(C_i, C_j)$ is a polysized formula (can be written down in polytime) if $M'$ has only polyspace.

QBF!

B: What is the size of

$\exists C_1, C_2, \dots : \varphi_{M',0}(C_i, C_j) \wedge \varphi_{M',0}(C_j, C_k) \cdots$




**Fact:** (see Pset)

$\varphi_{M',i}(C_1, C_2) = \exists C' \forall D_1, D_2 \left( (D_1 = C_1 \wedge D_2 = C') \vee (D_1 = C' \wedge D_2 = C_2) \right) \rightarrow \varphi_{M',i-1}(D_1, D_2)$

```
PrintQBF(M', i, C1, C2)
  · If i = 0:
      Return "B(C1,C2)"
  · Return "∃ C' ∀ D1,D2 (D1=C1 ∧ D2=C') ∨ (D1=C' ∧ D2=C2) → PrintQBF(M', i-1, D1, D2)"
```

all configs have input $x_1$, R/W size $S$

$T(S,i)$; $O(S)$; $O(S^2)$; $O(S)$; $T(S,i-1)$

```
f_MakeQBF(x, M', S)
  · Create Cstart, Cstart,x
  · g = PrintQBF(M', O(S), Cstart,x, Caccept)
  · Clean up g (move all quantifiers to front)
  · Return g.
```

← description of M's transition function

$O(S)$; $O(S^2)$; $O(S^3)$; $O(S^3)$; $O(S^2)$

Polytime ⇒ TQBF is PSPACE-hard

Runtime of f_MakeQBF?

Big Picture?

How would we have to change proof to show TQBF is NPSPACE Hard?