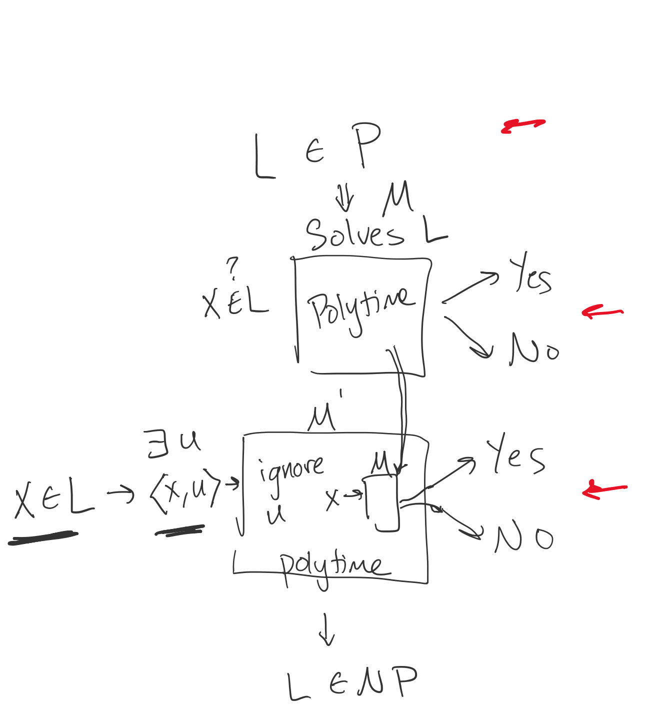
Thursday, February 24, 2022 1:27 PM

Goals

- Place NP in our map
- Define NP-Hard
- Prove Problems are NP-Hard

$P \subseteq NP$

Let $L \in P$. Then $\exists$ a polytime TM M such that $x \in L$ iff $M(x) = 1$.
(Let p be the constant function $p(n) = 1$.)
Let M' the TM that on input $\langle x, u \rangle$ for $u \in \{0,1\}^{p(n)}$ ignore u and apply M to x and output the result. Then if $x \in L$, then $\exists u \in \{0,1\}^{p(n)}$ such that $M'(x,u)$ accepts. If $x \notin L$ then $\forall u \in \{0,1\}^{p(n)}$, $M'(x,u)$ rejects. Since M' runs in polytime and u is polysized in x, we have $L \in NP$.



$NP \subseteq EXP$

Let $L \in NP$. Then there exists a TM M and a polynomial p s.t

$x \in L$ iff $\exists u \in \{0,1\}^{p(n)}$ s.t. $M(x,u) = 1$.

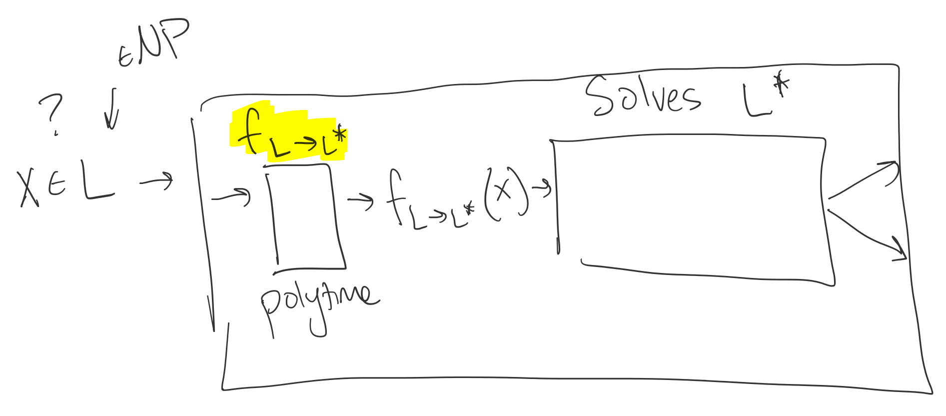
Let M' be the TM that on input x,

For every $u \in \{0,1\}^{p(n)}$
  Run $M(x,u)$
  If $M(x,u) = 1$, output 1 } Polytime } $\times 2^{p(n)}$ (n^d)

Output 0

Then $x \in L$ iff $M'(x)$ accepts.
M' runs in $(O(2^{n^d}))$ (exponential time).

def: $L^* \in$ NP-Hard if $\forall L \in NP$ $L \leq_p L^*$

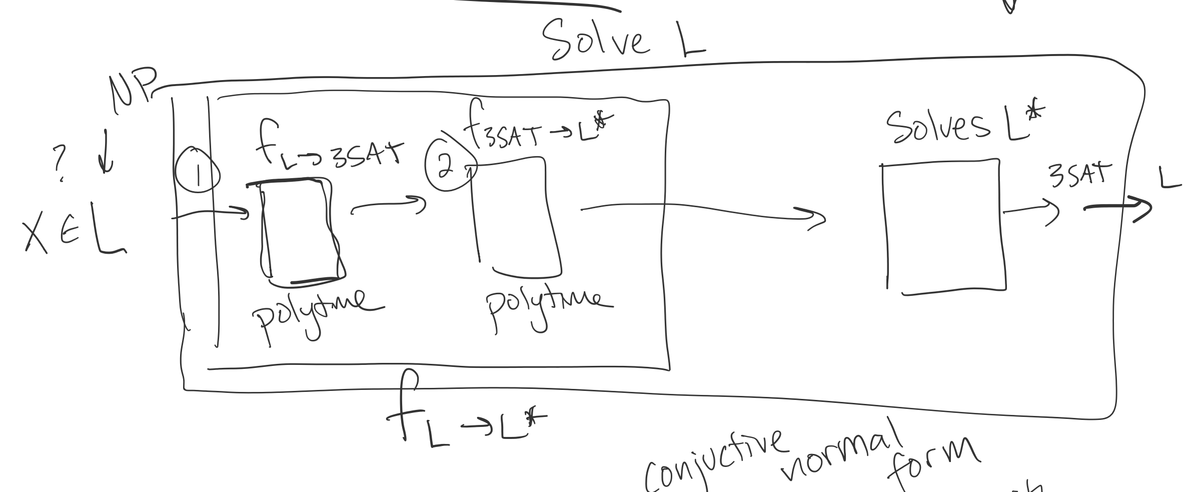


Idea:
- L^* is hard to solve
- Need powerful resources to solve L^* (at least NP type resources)

Idea: Reductions

1. 3SAT $\in$ NP-Hard (FACT)
2. 3SAT $\leq_p L^*$ (We prove)

} $\Rightarrow L^* \in$ NP-Hard



3SAT = $\{\langle x \rangle$: x is a CNF (conjuctive normal form) formula with a satisfying assignment and at most 3 literals in each clause$\}$

CNF Formula:

$$(u_1 \vee \neg u_2 \vee \neg u_4) \wedge (u_1 \vee u_3 \vee u_4) \wedge (\neg u_1 \vee u_2 \vee u_3) \ldots$$

(ORS: $\vee$; AND: $\wedge$; literals: $u_1, \neg u_2, \neg u_4$; clause: $(\neg u_1 \vee u_2 \vee u_3)$)

Assignment: $u_1 = \text{True}$
$u_2 = \text{False}$
$\vdots$
$u_n = \text{False}$

Group Work: Prove NP-Hard

- 01PROG = $\{X$: X is a set of inequalities with rational coefficients and has a satisfying assigment with assignments 0 or 1$\}$

ex: $\frac{1}{2}u_1 - \frac{3}{10}u_2 + \frac{8}{9}u_3 \geq 1/9$
$-\frac{5}{4}u_1 + \frac{2}{3}u_3 + \frac{1}{10}u_4 < -2/5$
$\vdots$

Assignment: $u_1 = 0$
$u_2 = 1$
$u_3 = 1$
$\vdots$

- CUBIC = $\{X$: X is a set of binary cubic equations with a satisfying assignment$\}$

ex: $u_1 \cdot u_1 \cdot u_1 + u_2 \cdot u_3 \cdot u_3 + u_1 \cdot u_2 \cdot u_3 = 0$
$u_1 \cdot u_2 \cdot u_4 + u_3 \cdot u_4 \cdot u_4 = 1$
$\vdots$

$u_1 = 0$
$u_2 = 1$
$u_3 = 1$
$\vdots$

Idea

$(u_1 \vee \neg u_2 \vee u_3) \Longrightarrow$ $\pm 1 u_1 + \pm 1 u_2 + \pm 1 u_3 \overset{?}{\gtrless}$ int. (b/t 1 and −2 incl.)
or
$u_1 \cdot u_2 \cdot u_3 = g$

Solution

- 01PROG:

3SAT		01PROG
$u_1 = \text{True}$	$\Longleftrightarrow$	$u_1 = 1$
$u_1 = \text{False}$	$\Longleftrightarrow$	$u_1 = 0$

$u_1 \vee u_2 \vee u_3 \longrightarrow u_1 + u_2 + u_3 \geq 1$

$\rightarrow$ $u_1 \vee u_2 \vee \neg u_3$ (T F F T) $\longrightarrow u_1 + u_2 - u_3 \geq 0$ (1 0 0 1)

$u_1 \vee \neg u_2 \vee \neg u_3 \longrightarrow u_1 - u_2 - u_3 \geq -1$

$\neg u_1 \vee \neg u_2 \vee \neg u_3 \longrightarrow -u_1 - u_2 - u_3 \geq -2$

- CUBIC

$u_1 = \text{True}$ in 3SAT $\rightarrow$ $u_1 = 1$, $\bar{u}_1 = 0$ in CUBIC

$u_1 = \text{False}$ in 3SAT $\rightarrow$ $u_1 = 0$, $\bar{u}_1 = 1$

$u_1 \neq \bar{u}_1 \Rightarrow u_1 u_1 u_1 + \bar{u}_1 \bar{u}_1 \bar{u}_1 = 1$

$u_1 \vee \neg u_2 \vee u_3 \longrightarrow$