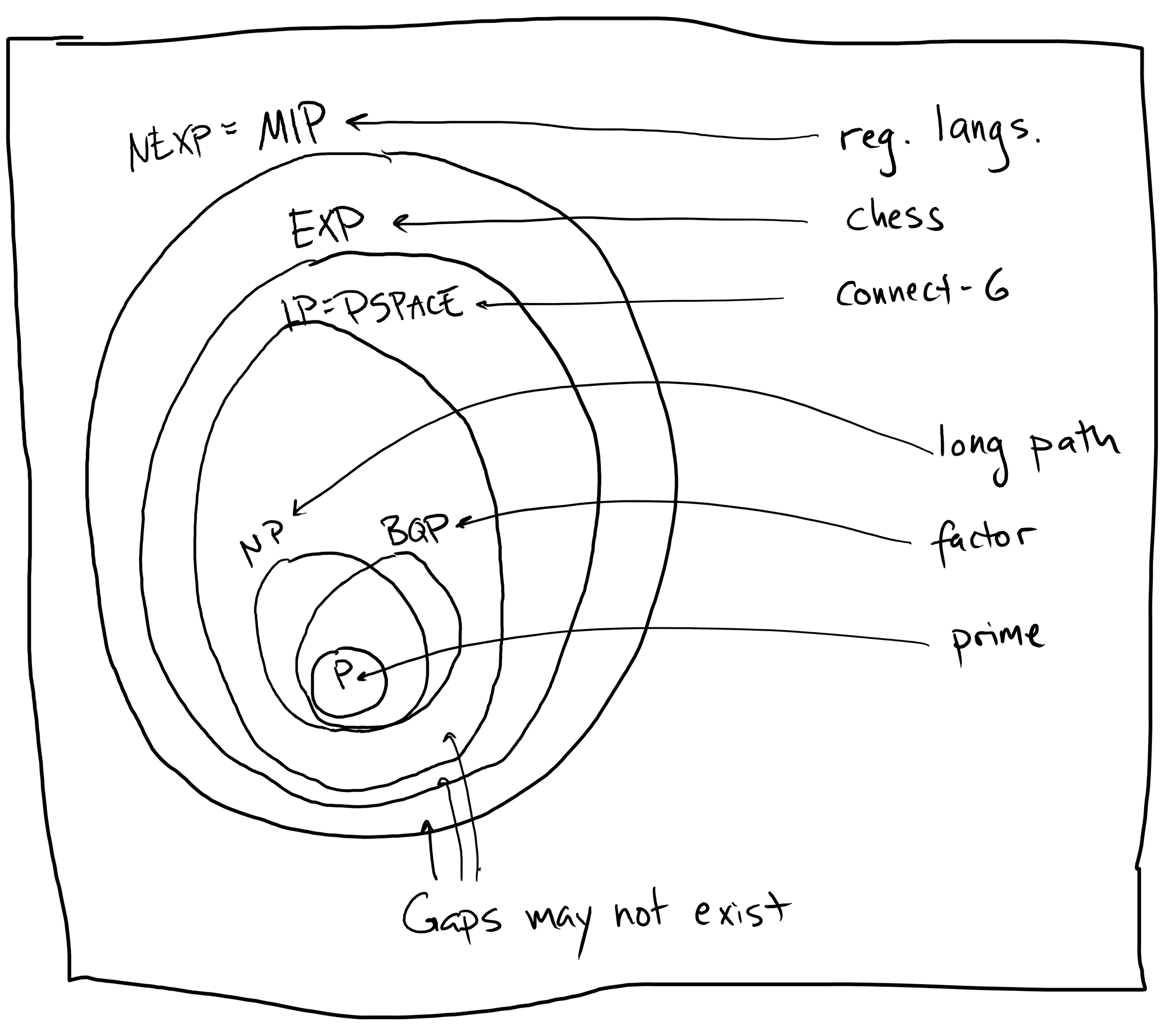
2s. Problems and Languages

Sunday, February 13, 2022 11:34 AM

Goals

- Mathematical definition of "Problem"
- Compare Decision and Function problems
- Understand connection between Decision Problems and Languages

Set of all ~~Problems~~: Languages ? Set of Sets



Example Problem: Addition

"Function Problem" ∞ possible outputs :(

- Input: $\langle x, y \rangle \to$ Input to TM: $\boxed{x \text{ in binary, } y \text{ in binary}}$
- Output: $x+y$ Output: $\boxed{x+y \text{ in binary}}$

Instead: "Decision Problem" = Yes/No problem = 0 or 1

Addition (Decision)

- Input: $\langle x, y, i \rangle$
- Output: $\begin{cases} 0 & \text{if } i^{th} \text{ bit of } x+y \text{ is } 0 \\ 1 & \text{"} \quad\quad\quad \text{"} \quad 1 \end{cases}$ function

Sets are simpler (mathematically) than functions

Addition (Decision) Language:

$$L = \{ \langle x, y, i \rangle : i^{th} \text{ bit of } x+y = 1 \}$$

Generally: $L = \{ \langle x \rangle : f(x) = 1 \}$

Alternate:

- Input: $\langle x, y, z \rangle$
- Output: $\begin{cases} 0 & \text{if } x+y < z \\ 1 & \text{if } x+y \geq z \end{cases}$ $\iff L = \{ \langle x, y, z \rangle : x+y \geq z \}$

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Group Work:

Function Problem: Which vertex in $G = (V, E)$ has the most edges?

- → Decision Problem?
- → Language?
- → Write language only using math

Input: $\langle G, v \rangle$ $G = (V, E)$

Output: $\begin{cases} 1 & \text{if } v \text{ has most edges in } G \\ 0 & \text{else} \end{cases}$

$$L = \{ \langle G, v \rangle : v \text{ has most edges in } G \}$$

$$L = \{ \langle G, v \rangle : v \in V \wedge \forall_{w \in V} \, |\{u : \{u, v\} \in E\}| \geq |\{u : \{u, w\} \in E\}| \}$$

Venn Diagram

Important notation / terminology

- "TM M decides a language L"

$\Updownarrow$

If $x \in L$, $M(x) = 1$ ⟺ Input Tape $\boxed{x}$

If $x \notin L$, $M(x) = 0$ Output Tape $\boxed{1}$ $\boxed{0}$

- "Input size" $= |x| = n$ = # of bits on input tape

Addition (Decision) Language:

$$L = \{ \langle x, y, i \rangle : i^{th} \text{ bit of } x+y = 1 \}$$

What is $|\langle x, y, i \rangle|$?

A) $O(1)$ B) $O(\log(x+y))$ C) $O(\log x + \log y)$ D) $O(\log(xy))$

$x = 13 \to \boxed{1\,1\,0\,1}$ (8 4 2 0)

$\lceil \log_2 13 \rceil$

$$O\left(\log x + \log y + \log(\log(x+y))\right)$$

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