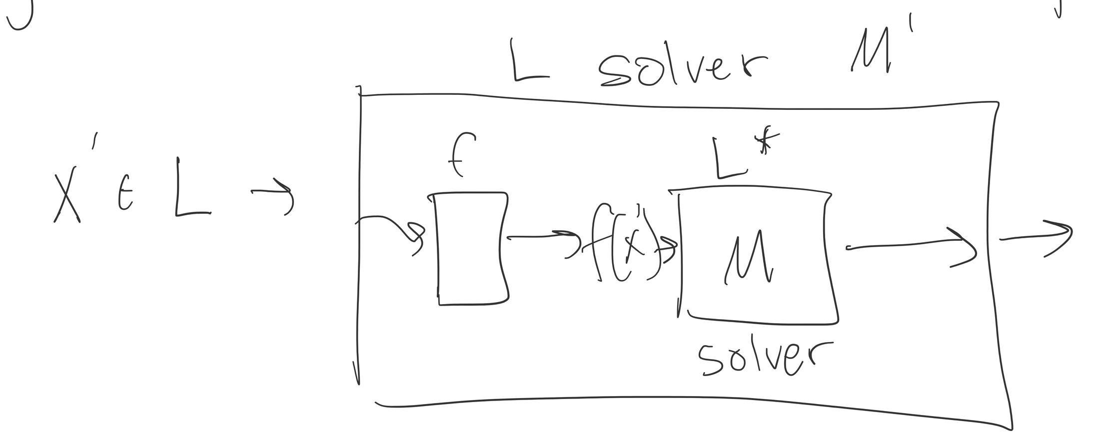
10s. Polynomial Hierarchy
Thursday, April 28, 2022
10:03 AM

Group Work

Let L^* be complete for the PH if $L^* \in PH$ and $\forall L \in PH$, $L \le_p L^*$. Prove that if such an L^* exists, the PH collapses

Pf: Since $L^* \in PH$, $\exists$ c s.t. $L^* \in \Sigma_c$. Let $L \in \Sigma_j$ for $j \ge c$. Then $L \le_p L^*$ so $\exists$ polynomial function f:



Let because $L^* \in \Sigma_c$, $\exists$ a polytime TM M and a polynomial q s.t

$$x \in L^* \text{ iff } \exists u_1 \in \{0,1\}^{q(|x|)} : \forall u_2 \in \{0,1\}^{q(|x|)} \cdots Q u_c \in \{0,1\}^{q(|x|)} \; M(x, u_1, \dots u_c) = 1$$

Let M' be the machine that on input $(x', u_1, \dots u_c)$, runs f on x' to get output $f(x')$, and then runs $M(f(x'), u_1, \dots u_c)$. M' runs in polynomial time, and by our reduction,

$$x' \in L \text{ iff } \exists u_1 \in \{0,1\}^{q(|x|)} : \forall u_2 \in \{0,1\}^{q(|x|)} \cdots Q u_c \in \{0,1\}^{q(|x|)} \; M'(x', u_1, \dots u_c) = 1$$

so $L \in \Sigma_c$, so $\Sigma_j \subseteq \Sigma_c$.

Goals:
- Write collapse proofs

Announcements
- Go/ace (anti-procrastination station)
- Final
- Keep demonstrating effort/learning

Theorem: If $P = NP$ then the PH Collapses (to 0^{th} level)

Pf: We will prove by induction on n that if $P = NP$ then $P = \Sigma_n$ and $P = \Pi_n$

Base Case: For $n = 1$, $\Sigma_1 = NP$. By our assumption, $P = NP$, so $P = \Sigma_1$. If $P = NP$, then $P = coNP$ (Pset 4 #4), so $P = \Pi_1$

Inductive Step: Let $k \ge 1$. Assume for induction that $P = \Pi_k = \Sigma_k$. Let $L \in \Sigma_{k+1}$. That means $\exists$ a polytime TM M and polynomial q s.t

$$x \in L \text{ iff } \exists u_1 \in \{0,1\}^{q(|x|)} : \forall u_2 \in \{0,1\}^{q(|x|)} \cdots Q u_{k+1} \in \{0,1\}^{q(|x|)} \; M(x, u_1, u_2 \dots u_{k+1}) = 1$$

Define L':

$$\langle x, u_1 \rangle \in L' \text{ iff } \forall u_2 \in \{0,1\}^{q(|x|)} \cdots Q u_{k+1} \in \{0,1\}^{q(|x|)} \; M(x, u_1, u_2 \dots u_{k+1}) = 1$$

By def of Π_k, we have $L' \in \Pi_k$. By our inductive assumption, $L' \in P$. That means $\exists$ a polytime TM M' s.t.

$$\langle x, u_1 \rangle \in L' \text{ iff } M'(x, u_1) = 1$$

Another way to define L:

$$x \in L \text{ iff } \exists u_1 : \boxed{\langle x, u_1 \rangle \in L'}$$

$$\parallel$$

$$\boxed{\forall u_2 \in \{0,1\}^{q(|x|)} \cdots Q u_{k+1} \in \{0,1\}^{q(|x|)} \; M(x, u_1, u_2 \dots u_{k+1}) = 1}$$

$$\parallel$$

$$\boxed{M'(x, u_1) = 1}$$

So $L \in NP$. But by our assumption, $P = NP$, so $L \in P$.
So $\Sigma_{k+1} \subseteq P$

Because $P = coP$, $\Pi_{k+1} \subseteq P$. □