CS333 - Problem Set 6

1. The Bloch sphere is a useful tool for visualizing single qubit states, gates, and measurements. In the next few questions, we'll investigate this tool.

   The most general way we can write a 2 qubit state (that fulfills the normalization condition), is

   $$\begin{pmatrix} e^{i\chi}\cos\theta \\ e^{i\xi}\sin\theta \end{pmatrix}, \tag{1}$$

   where χ (chi, "kai"), ξ (xi, "ksai", like "excise" but without the first e sound or last z sound), and θ (theta, "thay-tah," th sound like in thin) are any real numbers.

   However, in PS3 no. 4, we saw that we can multiply both parts of a quantum state by a global phase $e^{i\omega}$ and that has no physical effect on the state. We can use this phase freedom to always choose the $|0\rangle$ amplitude of the state to be real (i.e. to remove the $e^{i\chi}$ term). Thus we can always represent a single qubit state in the following way, using only two real parameters: θ and φ :

   $$|\psi(\theta,\varphi)\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix}. \tag{2}$$

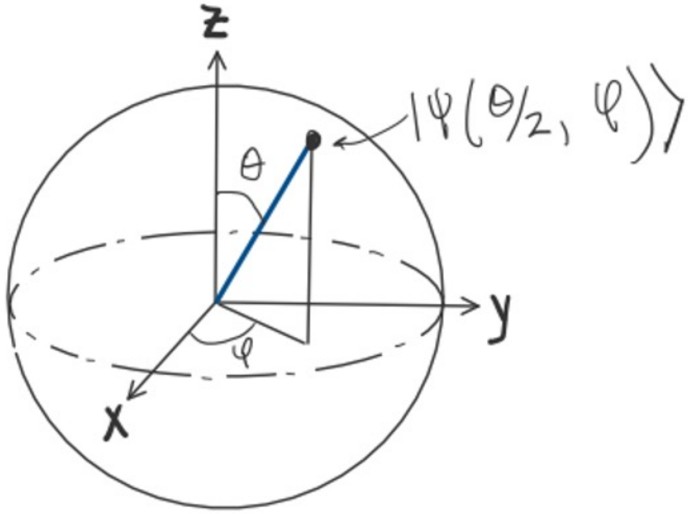
   We can further use the global phase freedom to choose to make $\cos(\theta)$ positive. Using that constraint and the fact that the functions are periodic, if we choose $\theta \in [0, \pi/2]$ and $\varphi \in [0, 2\pi)$, we can represent every possible single qubit state with vectors of the form of Eq. (2). (φ is caligraphic ϕ, which is phi, "fie").

   Next, note that we can label every single point on the surface of a sphere at radius 1 from the origin using two parameters: θ' and φ', where $\theta' \in [0, \pi]$ and $\varphi' \in [0, 2\pi)$. (Note θ' can have values between 0 and π, whereas θ can only have values between 0 and $\pi/2$). We do this by using θ' to represent the polar angle (the angle between the point in question and the north pole, like latitude) and using φ' to represent the azimuthal angle (longitude). (If you would like more info on this, please look up spherical coordinates. See also Fig. 1.)

   Because of this similarity in parameters between the qubit and the sphere, you can associate each physical qubit state with a unique point on the surface of a sphere. To do this, we identify the state $|\psi(\theta,\varphi)\rangle$ with the point on the sphere with polar angle $\theta' = 2\theta$ and azimuthal angle $\varphi' = \varphi$, as in the following diagram. (**Important:** the θ and θ' always differ from each other by a factor of 2, so be careful of this when moving from one representation to another.) When we are thinking of the sphere as a space where qubit states live, we call it the Bloch Sphere.

   (a) Explain why the vector **z** (north pole direction) corresponds to $|0\rangle$, and **-z** (south pole direction) corresponds to $|1\rangle$.

Figure 1: Bloch Sphere



(b) What qubit states do the vectors **x**, **y**, **-x**, and **-y** correspond to? (See figure. Note **y** is $90°$ from **x** on the equator.)

(c) What is the absolute value squared of the inner product of any two states that are at $90°$ from each other? $180°$ from each other? (You do not need to prove this result, although you can if you want! Instead extrapolate from examples.)

(d) What does a single qubit measurement correspond to on the Bloch Sphere? (Again you do not need to prove this, just extrapolate from examples.)

(e) Single qubit unitaries correspond to rotations of the Bloch sphere. (This intuitively makes sense because unitaries transform one state to another, and a rotation transforms one point on the sphere's surface to another.) A rotation is defined by two quantities: an axis of rotation, and an angle of rotation. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \tag{3}$$

is a rotation around the **z** axis (because this unitary doesn't change the state $|0\rangle$). In general, the direction of rotation is found using the "right hand rule:" point your right thumb in the direction of the rotation (in this case **z**), and then the rotation is in the direction your fingers move when you start with an open hand and then close it. To figure out the angle of rotation, we can test what happens to states on the effective equator of the rotation axis. For Eq. (3), because the rotation is about the north pole, the effective equator is the actual equator, and we can look at what this unitary does to a state on the equator. For example, $|+\rangle$ is on the equator, and we see that the unitary turns $|+\rangle$ into $|\leftarrow\rangle$. This is a $90°$ rotation, but in the opposite direction of the right hand rule, so this unitary corresponds to a rotation of $-\pi/2$ ($-90°$) of the Bloch sphere. (Or you could equally describe it as a $3\pi/2$ positive rotation.) Alternatively, we could describe this unitary as a rotation around the **-z** axis by an angle of $\pi/2$. (There are always two possible choices for the axis of rotation: a vector and it's negation on the sphere, however you need to make sure that the angle you choose is in the proper direction relative to the right hand rule.)

What is the axis of rotation and angle of rotation of the Bloch sphere corresponding to each of the following unitaries?

i. I

ii. X

iii. Z

iv. Y

v. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

vi. (Challenge) H

(f) (Challenge) Based on the examples in part (d), please extrapolate the connection between the eigenvectors and eigenvalues of a single qubit unitary, and the corresponding Bloch sphere rotation.

2. Consider the single-qubit operation $I - 2|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is any valid qubit state and I is the 2×2 identity matrix. (This unitary will be important for the quantum searching algorithm)

   (a) Show $I - 2|\psi\rangle\langle\psi|$ is a unitary operation.

   (b) Describe what U does.

3. The fields of quantum computing and computer science, and in fact U.S. society as a whole, are currently engaged in dicsussions about the use of language, especially perceived inclusive language vs. racist/sexist/ableist/etc. language. We are asking questions like: how can language choices foster inclusivity? if our language use changes, but the underlying systems of inequality do not, does it even matter? how do we navigate an evolving language landscape amidst cancel culture? what is the intersection of politics and language choices? how do we draw the line between acceptable and unacceptable language as language use evolves?

   In the field of quantum computing, there has recently been discussion around the term "quantum supremacy." Please read this blog post by Scott Aaronson for a fairly nuanced (although certainly opinionated) description of the discussion. In computer science, we have had similar discussions around terms like "master/slave" and "blacklist;" some of these discussions have resulted in widespread changes in our language use, others have not. I noticed a term in one of the optional readings from earlier in the semester that I don't think has been discussed but which could be: "man-in-the-middle attack."

   After reading the blog post, please write a short written reflection on some of the above questions (and any others that you find relevant) both in the context of "quantum supremacy" and more broadly in the field of computer science. Can you think of other examples of how language is evolving or might evolve in quantum computing and computer science? What is the significance of scientific communities (communities that we might expect to be primarily focused on doing science) engaging in this process of reflection and discussion of language use? We will spend some time in class discussing our reflections, so please come prepared to share some of your thoughts.