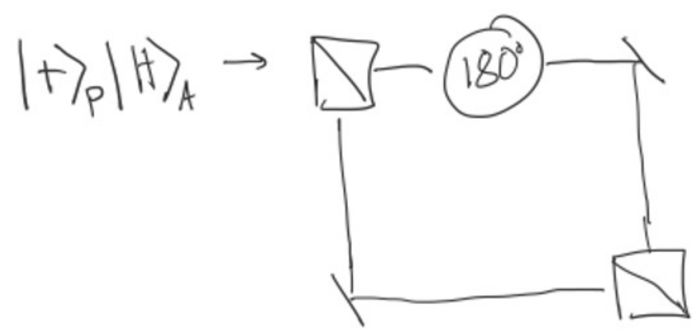
CS333 - Problem Set 5

Note: Problems 2 and 3 are somewhat challenging, in that they ask you to apply the skills we have been learning in ways that are somewhat different from the examples we've seen in class. We will go over questions you have about them in class on April 1, so please post questions you have after doing your self-assessment to the PS5 Discussion channel (or like others' questions).



1. Consider the interferometer in the figure above.

   (a) What will be the polarization(s)/direction(s) of the photon that exits the interferometer? Note that a 180° lens turns $|0\rangle \to -|0\rangle$

   (b) In Problem Set 3, no. 4, you showed that if you multiply a state by -1, it is not physically different from a state that is not multiplied by -1. (Note that $-1 = e^{i\pi}$.) However, in part (a) of this problem, we see that multiplying by -1 does change the physical outcome relative to the case with no lens in the interferometer. (What happens with no lens?) What is different here from PS3 no. 4?

2. **[Moved to a future problem set]** The Bloch sphere is a useful tool for visualizing single qubit states, gates, and measurements. In the next few questions, we'll investigate this tool.

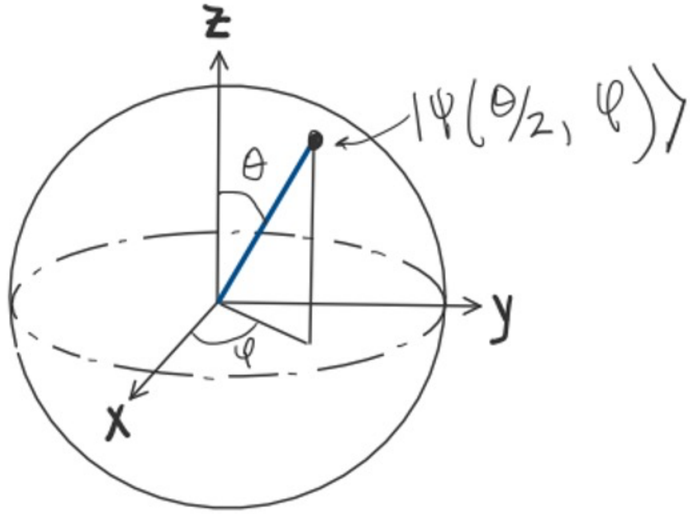
   The most general way we can write a 2 qubit state (that fulfills the normalization condition), is

$$\begin{pmatrix} e^{i\chi}\cos\theta \\ e^{i\xi}\sin\theta \end{pmatrix}, \tag{1}$$

   where χ (chi, "kai"), ξ (xi, "ksai", like "excise" but without the first e sound or last z sound), and θ (theta, "thay-tah," th sound like in thin) are any real numbers.

   However, in PS3 no. 4, we saw that we can multiply both parts of a quantum state by a "phase" $e^{i\omega}$ and that has no physical effect on the state. (A phase is what we call a complex number whose absolute value squared is 1.) So we can use this phase freedom to always

Figure 1: Bloch Sphere



choose the $|0\rangle$ amplitude of the state to be real. Thus we can always represent a single qubit state in the following way, using only two real parameters: θ and φ :

$$|\psi(\theta, \varphi)\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix}. \tag{2}$$

We can further use the global phase freedom to choose to make $\cos(\theta)$ positive. Using that constraint and the fact that the functions are periodic, we can choose where $\theta \in [0, \pi/2]$ and $\varphi \in [0, 2\pi)$, and represent every possible single qubit state with Eq. (2). (φ is caligraphic ϕ, which is phi, "fie").

Next, note that we can label every single point on the surface of a sphere at radius 1 from the origin using two parameters: θ' and φ', where $\theta' \in [0, \pi]$ and $\varphi' \in [0, 2\pi)$. (Note θ' can have values between 0 and π, whereas θ can only have values between 0 and $\pi/2$). We do this by using θ' to represent the polar angle (the angle between the point in question and the north pole, like latitude) and using φ' to represent the azimuthal angle (longitude). (If you would like more info on this, please look up spherical coordinates. See also Fig. 1.)

Because of this similarity in parameters between the qubit and the sphere, you can associate each qubit state with a unique point on the surface of a sphere. To do this, we identify the state $|\psi(\theta, \varphi)\rangle$ with the point on the sphere with polar angle $\theta' = 2\theta$ and azimuthal angle $\varphi' = \varphi$, as in the following diagram. (**Important:** the θ and θ' always differ from each other by a factor of 2, so be careful of this when moving from one representation to another.) When we are thinking of the sphere as a space where qubit states live, we call it the Bloch Sphere.

(a) Explain why the vector **z** (north pole direction) corresponds to $|0\rangle$, and **-z** (south pole direction) corresponds to $|1\rangle$.

(b) What qubit states do the vectors **x**, **y**, **-x**, and **-y** correspond to? (See figure. Note **y** is 90° from **x** on the equator.)

(c) What is the absolute value squared of the inner product of any two states that are at 90° from each other? 180° from each other? (You do not need to prove this result, although you can if you want! Instead extrapolate from examples.)

(d) What does a single qubit measurement correspond to on the Bloch Sphere? (Again you do not need to prove this, just extrapolate from examples.)

(e) Single qubit unitaries correspond to rotations of the Bloch sphere. (This intuitively makes sense because unitaries transform one state to another, and a rotation transforms one point on the sphere's surface to another.) A rotation is defined by two quantities: an axis of rotation, and an angle of rotation. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \tag{3}$$

is a rotation around the **z** axis (because this unitary doesn't change the state $|0\rangle$). In general, the direction of rotation is found using the "right hand rule:" point your right thumb in the direction of the rotation (in this case **z**), and then the rotation is in the direction your fingers move when you start with an open hand and then close it. To figure out the angle of rotation, we can test what happens to states on the effective equator of the rotation axis. For Eq. (3), because the rotation is about the north pole, the effective equator is the actual equator, and we can look at what this unitary does to a state on the equator. For example, $|+\rangle$ is on the equator, and we see that the unitary turns $|+\rangle$ into $|\leftarrow\rangle$. This is a 90° rotation, but in the opposite direction of the right hand rule, so this unitary corresponds to a rotation of $-\pi/2$ ($-90°$) of the Bloch sphere. (Or you could equally describe it as a $3\pi/2$ positive rotation.) Alternatively, we could describe this unitary as a rotation around the **-z** axis by an angle of $\pi/2$. (There are always two possible choices for the axis of rotation: a vector and it's negation on the sphere, however you need to make sure that the angle you choose is in the proper direction relative to the right hand rule.)

What is the axis of rotation and angle of rotation of the Bloch sphere corresponding to each of the following unitaries?

i. I

ii. X

iii. Z

iv. Y

v. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

vi. (Challenge) H

(f) (Challenge) Based on the examples in part (d), please extrapolate the connection between the eigenvectors and eigenvalues of a single qubit unitary, and the corresponding Bloch sphere rotation.

3. In this problem, you will explore one of the most famous applications of entangled states: quantum teleportation. Ahsoka has a qubit A_1 in the state $a|0\rangle + b|1\rangle$ that she would like to send to Boba Fett. Ahsoka and Boba can communicate classically (e.g. they can talk on the phone) but they don't have a quantum channel, so she can't send the qubit to Boba. You

might be thinking that one solution is for Ahsoka to just tell Boba the values of a and b, so then he can just create his own copy of the qubit, but let's assume that Ahsoka doesn't know what a and b are; Grogu gave her the qubit without telling her its state. (Yes - I am aware that I am mixing my sci-fi metaphors.)

What Ashoka and Boba do share is an entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_2B}$, where A_2 is Ahsoka's qubit and B is Boba's qubit. So altogether, they have the state

$$(a|0\rangle + b|1\rangle)_{A_1} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{A_2B} \tag{4}$$

$$= \frac{1}{\sqrt{2}}(a|000\rangle_{A_1A_2B} + a|011\rangle_{A_1A_2B} + b|100\rangle_{A_1A_2B} + b|111\rangle_{A_1A_2B} \tag{5}$$

where the second expression comes from distributing the tensor product, and then using our compressed tensor product notation for standard basis states. (We are now considering 3-qubit states, which are represented using 8-dimensional vectors.)

Here are the steps of the protocol:

(a) Ahsoka applies the gate that acts on standard basis states as:

$$|00\rangle_{A_1A_2} \rightarrow \frac{1}{\sqrt{2}}\left(|00\rangle_{A_1A_2} + |01\rangle_{A_1A_2}\right) \tag{6}$$

$$|01\rangle_{A_1A_2} \rightarrow \frac{1}{\sqrt{2}}\left(|10\rangle_{A_1A_2} + |11\rangle_{A_1A_2}\right) \tag{7}$$

$$|10\rangle_{A_1A_2} \rightarrow \frac{1}{\sqrt{2}}\left(|10\rangle_{A_1A_2} - |11\rangle_{A_1A_2}\right) \tag{8}$$

$$|11\rangle_{A_1A_2} \rightarrow \frac{1}{\sqrt{2}}\left(|00\rangle_{A_1A_2} - |01\rangle_{A_1A_2}\right) \tag{9}$$

$$\tag{10}$$

to her two qubits A_1 and A_2, while Boba does nothing.

(b) Ahsoka measures A_1 and A_2 in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. (Partial measurement of A_1 and A_2).

(c) Ahsoka sends the result of her measurement to Boba using the classical channel. (Ahsoka sends the message "00", "01", "10", or "11".)

(d) Bob does the following

- If he gets the message "00", he applies the gate I to his qubit B. I acts on standard basis states in the following way:

$$|0\rangle \rightarrow |0\rangle \tag{11}$$

$$|1\rangle \rightarrow |1\rangle. \tag{12}$$

- If he gets the message "01", he applies the gate Z to his qubit B. Z acts on standard basis states in the following way:

$$|0\rangle \rightarrow |0\rangle \tag{13}$$

$$|1\rangle \rightarrow -|1\rangle. \tag{14}$$

- If he gets the message "10", he applies the gate X to his qubit B. X acts on standard basis states in the following way:

$$|0\rangle \rightarrow |1\rangle \tag{15}$$
$$|1\rangle \rightarrow |0\rangle. \tag{16}$$

- If he gets the message "11", he applies the gate Y to his qubit B. Y acts on standard basis states in the following way:

$$|0\rangle \rightarrow i|1\rangle \tag{17}$$
$$|1\rangle \rightarrow -i|0\rangle. \tag{18}$$

Show that Bob will always possess the state $a|0\rangle_B + b|1\rangle_B$ at the end of the protocol.