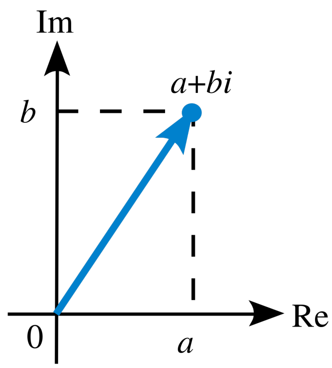
CS333 - Math for Quantum

1 Complex Numbers

A complex number is written as $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. It is useful to think about complex numbers as points on a plane:



If $x = a + bi$, then the *complex conjugate* of x, denoted x^*, is $a - bi$. If w and x are complex numbers, then

$$\begin{aligned}(wx)^* &= w^* x^* \\ (w + x)^* &= w^* + x^* \end{aligned} \tag{1}$$

Question: If we think of complex numbers as points on a plane, and conjugation as a function that takes a complex number to its conjugate, how does the operation of conjugation transform points on a plane?
Solution: It acts as a reflection over the x-axis (the real axis). (A reflection flips points over an axis of reflection.)

For $w \in \mathbb{R}$, Euler's formula is the following:

$$e^{iw} = \cos(w) + i \sin(w). \tag{2}$$

This formula lets us use a complex exponential to represent complex numbers.

Question: If $w \in \mathbb{R}$, which of the following is the complex conjugate of e^{iw}?

$$e^{-iw}, \qquad \cos(w) - i\sin(w), \qquad -\cos(w) + i\sin(w), \qquad e^{w} \tag{3}$$

Solution: e^{-ix} and $\cos(x) - i\sin(x)$

If $x = a + bi$, we denote the magnitude of x as $|x|$, where $|x| = \sqrt{xx^*}$.

Question: Explain why $|e^{ix}| = 1$ twice. The first time, use the e^{ix} representation, and the second use the $\cos(x) + i\sin(x)$ representation. Note: $a^0 = 1$ for any non-zero number a. Also, note: $a^b a^c = a^{b+c}$. Finally, note: $\cos^2(a) + \sin^2(a) = 1$ for any real number a.
Solution: First, we have $|e^{ix}| = \sqrt{e^{ix}e^{-ix}} = \sqrt{e^{ix-ix}} = \sqrt{e^0} = \sqrt{1} = 1$. Then, we also have $|e^{ix}| = |\cos(x) + i\sin(x)| = \sqrt{(\cos(x) + i\sin(x))(\cos(x) - i\sin(x))} = \sqrt{\cos^2(x) + \sin^2(x)} = \sqrt{1} = 1$.

Question: What is a simpler expression for $e^{i3\pi/2}$, and where does it appear as a point on the plane?
Solution: $e^{i3\pi/2} = -i$, and it is the a point on the negative y-axis a distance 1 from the origin.

2 Vector Spaces

We will deal with vector spaces $\mathbb{C}^d$. $\mathbb{C}^d$ is the set of column vectors of length d (dimension d) whose elements are complex numbers. So for example

$$\begin{pmatrix} 1 \\ i \\ e^{2i} \end{pmatrix} \in \mathbb{C}^3. \tag{4}$$

If $\mathbf{x} \in \mathbb{C}^d$, then the conjugate transpose of $\mathbf{x}$, denoted by $\mathbf{x}^\dagger$, is the d-dimensional row vector where the jth element of $\mathbf{x}^\dagger$ is the complex conjugate of the jth element of $\mathbf{x}$. For example, if

$$\mathbf{x} = \begin{pmatrix} 1 \\ i \\ e^{2i} \end{pmatrix} \qquad \text{then} \qquad \mathbf{x}^\dagger = \begin{pmatrix} 1, & -i, & e^{-2i} \end{pmatrix}. \tag{5}$$

More generally, if you have a matrix $\mathbf{A}$, then $\mathbf{A}^\dagger$ denotes the conjugate transpose of $\mathbf{A}$, where you take the transpose of the matrix, and then take the complex conjugate of each element. For example

$$\begin{pmatrix} 1, & 2, & 3 \\ -1i, & -2i, & -3i \end{pmatrix}^\dagger = \begin{pmatrix} 1, & 1i \\ 2, & 2i \\ 3, & 3i \end{pmatrix}$$

Given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^d$, we can take the inner product $\mathbf{y}^\dagger \mathbf{x}$ by doing matrix multiplication between $\mathbf{y}^\dagger$ and $\mathbf{x}$. That is: $\mathbf{y}^\dagger \mathbf{x} = \sum_{i=1}^d y_i^* x_i$, where x_i is the ith element of $\mathbf{x}$ and y_i is the ith element of $\mathbf{y}$. For example, if $\mathbf{x}$ is as above, and $\mathbf{y} = \begin{pmatrix} i \\ 1+i \\ 2 \end{pmatrix}$, then

$$\begin{aligned} \mathbf{y}^\dagger \mathbf{x} = \begin{pmatrix} -i, & 1-i, & 2 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ e^{2i} \end{pmatrix} = -i \times 1 + (1-i) \times i + 2 \times e^{2i} &= -i + i + 1 + 2e^{2i} \\ &= 1 + e^{2i}. \end{aligned} \tag{6}$$

Question: If $\mathbf{x} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, what is $\mathbf{y}^\dagger \mathbf{x}$? What is $\mathbf{x}^\dagger \mathbf{y}$?
Solution: $\mathbf{y}^\dagger \mathbf{x} = \mathbf{x}^\dagger \mathbf{y} = 0$.

Question: If $\mathbf{x}, \mathbf{y}$ are any vectors in $\mathbb{C}^d$, explain why $\left(\mathbf{y}^\dagger \mathbf{x}\right)^* = \mathbf{x}^\dagger \mathbf{y}$.
Solution: Using the rules for adding and multiplying complex numbers, we have

$$\left(\mathbf{y}^\dagger \mathbf{x}\right)^* = \left(\sum_{i=1}^d y_i^* x_i\right)^* = \sum_{i=1}^d \left(y_i^* x_i\right)^* = \sum_{i=1}^d \left(y_i^*\right)^* \left(x_i\right)^* = \sum_{i=1}^d y_i x_i^* = \mathbf{x}^\dagger \mathbf{y}. \tag{7}$$

Question: Show that the inner product follows the distributive property. That is, if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^d$, explain why $\mathbf{z}^\dagger \left(\mathbf{x} + \mathbf{y}\right) = \mathbf{z}^\dagger \mathbf{x} + \mathbf{z}^\dagger \mathbf{y}$.
Solution: Using the definition of inner product, we have

$$\mathbf{z}^\dagger \left(\mathbf{x} + \mathbf{y}\right) = \sum_{i=1}^d z_i^* \left(x_i + y_i\right) = \sum_{i=1}^d z_i^* x_i + z_i^* y_i = \left(\sum_{i=1}^d z_i^* x_i\right) + \left(\sum_{j \in d} z_j^* y_j\right) = \mathbf{z}^\dagger \mathbf{x} + \mathbf{z}^\dagger \mathbf{y}. \tag{8}$$

A *basis* for a vector space $\mathbb{C}^d$ is a set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_d\}$ such that for every vector

$\mathbf{x} \in \mathbb{C}^d$, there is a unique set of complex numbers $\{a_1, \dots, a_d\}$ such that

$$\mathbf{x} = \sum_{j=1}^{d} a_j \mathbf{v}_j. \tag{9}$$

If you have a set of d vectors $\{\mathbf{v}_1, \dots \mathbf{v}_d\}$ each in $\mathbb{C}^d$ such that

$$\mathbf{v}_j^\dagger \mathbf{v}_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases} \tag{10}$$

then they form a basis for $\mathbb{C}^d$. We call such a basis an *orthonormal basis*. Orthonormal combines the words "orthogonal" which refers to two vectors whose inner product is orthogonal, and "normal" which refers to a vector whose inner product with itself is 1.

For example you can verify that:

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{11}$$

from an orthonormal basis for $\mathbb{C}^2$ because there are two of them, and they satisfy Eq. (10). In quantum computing, we will exclusively deal with orthonormal bases.

Given a vector $\mathbf{x} \in \mathbb{C}^d$, it is often helpful to write that vector in terms of a given orthonormal basis $\{\mathbf{v}_1, \dots \mathbf{v}_d\}$. In other words, we would like to find the complex numbers a_j as in Eq. (9). When $\{\mathbf{v}_1, \dots \mathbf{v}_d\}$ is an orthonormal basis, finding this decomposition is fairly straightforward. We can apply $\mathbf{v}_i^\dagger$ to both sides of equation Eq. (9):

$$\begin{aligned} \mathbf{v}_i^\dagger \mathbf{x} &= \mathbf{v}_i^\dagger \left(\sum_{j=1}^{d} a_j \mathbf{v}_j \right) \\ &= \sum_{j=1}^{d} a_j \mathbf{v}_i^\dagger \mathbf{v}_j \\ &= a_i \end{aligned} \tag{12}$$

where the second line comes from the distributive property, and the final line comes from Eq. (10).

Question: Consider the orthonormal basis $\{\mathbf{v_1}, \mathbf{v_2}\}$ for $\mathbb{C}^2$ where $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Write $\mathbf{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ in this basis. (In other words, write $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ for some complex numbers a_1 and a_2.)
Solution: We have

$$\begin{aligned} \mathbf{v}_1^\dagger \mathbf{x} &= \frac{1}{2}(1+i) \\ \mathbf{v}_2^\dagger \mathbf{x} &= \frac{1}{2}(1-i) \end{aligned} \tag{13}$$

so

$$\mathbf{x} = \frac{1}{2}(1+i)\mathbf{v}_1 + \frac{1}{2}(1-i)\mathbf{v}_2. \tag{14}$$

Let $\mathbb{C}^{n\times m}$ denote the set of matrices with n rows and m columns and complex elements. Let $\mathbf{A} \in \mathbb{C}^{n\times m}$ and $\mathbf{B} \in \mathbb{C}^{p\times q}$. Then the *tensor product* (technically the Kronecker product) of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$. If the element in the ith row and jth column of $\mathbf{A}$ is A_{ij}, then

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_{11}\mathbf{B}, & A_{12}\mathbf{B}, & \dots & A_{1m}\mathbf{B} \\ A_{21}\mathbf{B}, & A_{22}\mathbf{B}, & \dots & A_{2m}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}\mathbf{B}, & A_{n2}\mathbf{B}, & \dots & A_{nm}\mathbf{B} \end{pmatrix}.$$

For example,

$$\begin{pmatrix} 1, & 2, & 3 \\ -1, & -2, & -3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & 2\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & 3\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \\ -1\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & -2\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & -3\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1i & 2i & 3i \\ 0 & 0 & 0 \\ -1 & -2 & -3 \\ -1i & -2i & -3i \end{pmatrix}.$$

Question: If $\mathbf{A}$ is an $n \times m$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, what are the dimensions of $\mathbf{A} \otimes \mathbf{B}$?
Solution: $A \otimes B$ will be an $np \times mq$ matrix. This is because we will have m copies of B in the horizontal direction, and since each copy of B itself takes up q columns, the new matrix will have mq columns. Then in the vertical direction, there are n copies of B, and each copy has p rows, for np total rows.

The tensor product has the following properties:

$$
\begin{aligned}
x \otimes y &= xy && (15)\\
\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} && (16)\\
(\mathbf{B} + \mathbf{C}) \otimes \mathbf{A} &= \mathbf{B} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{A} && (17)\\
(\mathbf{A} \otimes \mathbf{B})^\dagger &= \mathbf{A}^\dagger \otimes \mathbf{B}^\dagger \qquad \text{Note! the order stays the same!} && (18)\\
(\mathbf{A} \otimes \mathbf{B}) \cdot (\mathbf{C} \otimes \mathbf{D}) &= \mathbf{A} \cdot \mathbf{C} \otimes \mathbf{B} \cdot \mathbf{D}, && (19)
\end{aligned}
$$

where $\cdot$ denotes regular matrix multiplication. (The first line means that if you just have numbers rather than matrices, the tensor product is just the regular product.)

Question: Consider the orthonormal basis $\{\mathbf{v_1}, \mathbf{v_2}\}$ for $\mathbb{C}^2$ where $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Show that $\{\mathbf{v_1} \otimes \mathbf{v_1}, \mathbf{v_1} \otimes \mathbf{v_2}, \mathbf{v_2} \otimes \mathbf{v_1}, \mathbf{v_2} \otimes \mathbf{v_2}\}$ is an orthonormal basis for $\mathbb{C}^4$.

Solution: Using the definition of tensor product, we have that

$$
\{\mathbf{v_1} \otimes \mathbf{v_1}, \mathbf{v_1} \otimes \mathbf{v_2}, \mathbf{v_2} \otimes \mathbf{v_1}, \mathbf{v_2} \otimes \mathbf{v_2}\} = \left\{ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\} \tag{20}
$$

You can check that Eq. (10) is satisfied by these vectors, and since there are 4 of them, they form an orthonormal basis for $\mathbb{C}^4$.

Question: In this problem, we'll show that the previous problem generalizes. Suppose we have an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ for $\mathbb{C}^d$, and an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_f\}$ for $\mathbb{C}^f$. Show that the set of vectors consisting of all possible pairs of tensor products $\mathbf{v}_i \otimes \mathbf{u}_j$ form an orthonormal basis, and say which space they form a basis for. (Use the tensor product properties!)

Solution: Since there are d vectors in the first set and f vectors in the second set, the number of possible pairs of vectors is df. Now if we take the inner product of two of the vectors, we have, using Eq. (18)

$$(\mathbf{v}_i \otimes \mathbf{u}_j)^\dagger \cdot (\mathbf{v}_k \otimes \mathbf{u}_l) = (\mathbf{v}_i^\dagger \otimes \mathbf{u}_j^\dagger) \cdot (\mathbf{v}_k \otimes \mathbf{u}_l). \tag{21}$$

Using Eq. (19), we have

$$(\mathbf{v}_i^\dagger \otimes \mathbf{u}_j^\dagger) \cdot (\mathbf{v}_k \otimes \mathbf{u}_l) = (\mathbf{v}_i^\dagger \mathbf{v}_k) \otimes (\mathbf{u}_j^\dagger \mathbf{u}_l). \tag{22}$$

Because $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_f\}$ are orthonormal bases, we will get terms that are either $0 \otimes 0 = 0$, $0 \otimes 1 = 0$, $1 \otimes 0 = 0$ or $1 \otimes 1 = 1$. The only time we get 1 is when $i = k$ and $j = l$, which is when we take the inner product of a basis vector with itself, and otherwise we get 0. Note that these vectors are elements of $\mathbb{C}^{df}$, so since they fulfil Eq. (10), they form an orthonormal basis for $\mathbb{C}^{df}$.