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- $R(1 \text{ qbit})$
  - ∞ cbits (need to describe 2 real $\measuredangle$'s of Bloch sphere)
  - 1 qbit
  - 1 ebit + 2 cbits = "Teleportation"

Strategy

1. A & B start with $\underbrace{|\psi\rangle_{A_1}}_{\text{state to send}} \underbrace{|\beta_{00}\rangle_{A_2 B}}_{\text{ebit shared}}$ (1 ebit)
2. Alice measures A_1 and A_2 (this destroys entanglement) using Bell Basis
3. Alice sends outcome of measurement to Bob (2 cbits)
4. Bob applies a unitary to his system B based on Alice's cbits

New skill: what happens when only part of system is measured?

Recall: If both measured, effective measurement is

$$M_A \otimes M_B$$

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Partial Measurement

Let $|\psi\rangle_{AB}$ be a state on systems A (N_A-dim) and B (N_B-dim)

- Alice measures with $M_A = \{|\phi_1\rangle, \ldots |\phi_{N_A}\rangle\}$,
- Bob does not measure → If Alice gets outcome $|\phi_i\rangle$, what happens to Bob's state?

1. It is always possible to write (uniquely)

$$|\psi_{AB}\rangle = \sum_{i=0}^{N_A} a_i |\phi_i\rangle_A |\chi_i\rangle_B$$

Complex number, where $\sum |a_i|^2 = 1$

element of M_A

N_B dimensional properly normalized state

2. Then Alice gets outcome $|\phi_i\rangle$ with probability $|a_i|^2$, and in this case, the system is left in the state

$$|\phi_i\rangle_A |\chi_i\rangle_B$$

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How to change basis

Orthonormal basis $\{|\psi_i\rangle\}$

$$\underset{\uparrow \atop \text{identity matrix}}{\mathbb{I}} = \sum_i |\psi_i\rangle\langle\psi_i|$$

$$\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a^* & b^* \end{pmatrix} = \begin{pmatrix} aa^* & ab^* \\ a^*b & bb^* \end{pmatrix}$$

Ketbra $\Longrightarrow$ matrix

Write $|\phi\rangle$ using $\{|\psi_i\rangle\}$ basis:

$$|\phi\rangle = \mathbb{I}|\phi\rangle = \sum_i |\psi_i\rangle\langle\psi_i||\phi\rangle = \sum_i \overbrace{\left(\langle\psi_i|\phi\rangle\right)}^{\text{amplitude}} \overbrace{|\psi_i\rangle}^{\text{basis state}}$$

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ex:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

Measure A in $\{|+\rangle, |-\rangle\}$

- What outcomes occur with what probability?
- What happens to B system?

1. Put system A into $\{|+\rangle, |-\rangle\}$ basis:

$$I_{AB} = I_A \otimes I_B = \left(|+\rangle\langle+| + |-\rangle\langle-|\right) \otimes I_B$$

$$I_{AB}|\psi\rangle = \left(|+\rangle\langle+| + |-\rangle\langle-|\right)_A \otimes I_B \left(\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\right)_{AB}$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle\underbrace{\langle+|0\rangle_A}|0\rangle_B + |+\rangle\underbrace{\langle+|1\rangle_A}|1\rangle_B + |-\rangle\underbrace{\langle-|0\rangle_A}|0\rangle_B + |-\rangle\underbrace{\langle-|1\rangle_A}|1\rangle_B\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+\rangle|0\rangle + \frac{1}{\sqrt{2}}|+\rangle|1\rangle + \frac{1}{\sqrt{2}}|-\rangle|0\rangle + \frac{-1}{\sqrt{2}}|-\rangle|1\rangle\right)$$

$$= \frac{1}{2}\left(|+\rangle\left(|0\rangle + |1\rangle\right) + |-\rangle\left(|0\rangle - |1\rangle\right)\right)$$

not normalized

$$= \frac{1}{\sqrt{2}}\left(|+\rangle\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + |-\rangle\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle + |-\rangle|-\rangle\right)$$

$|+\rangle$: $Pr\left(\frac{1}{2}\right)$ $B \rightarrow |+\rangle$

$|-\rangle$: $Pr\left(\frac{1}{2}\right)$ $B \rightarrow |-\rangle$

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Strategy

state to send $= a|0\rangle + b|1\rangle$

ebit shared

1. A & B start with $|\psi\rangle_{A_1} |\beta_{00}\rangle_{A_2 B}$ (1 ebit)
2. Alice measures A_1 and A_2 (this destroys entanglement) using Bell Basis
3. Alice sends outcome of measurement to Bob (2 cbits)
4. Bob applies a unitary to his system B based on Alice's cbits

What state does Bob end up with in each case?

Which unitary should Bob apply for each outcome?

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1. Alice measures qubits A_1 & A_2 in Bell basis (recall strategy from composite systems)

$$|\psi\rangle_{A_1}|\beta_{00}\rangle_{A_2B} = (a|0\rangle + b|1\rangle)\otimes\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$= \frac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\right)_{A_1A_2B}$$

$$\begin{aligned}
|\psi\rangle|\beta_{00}\rangle = \;& + |\beta_{00}\rangle\langle\beta_{00}|_{A_1A_2}\otimes I_B \;|\psi\rangle|\beta_{00}\rangle \\
& + |\beta_{01}\rangle\langle\beta_{01}|_{A_1A_2}\otimes I_B \;|\psi\rangle|\beta_{00}\rangle \\
& + |\beta_{10}\rangle\langle\beta_{10}|_{A_1A_2}\otimes I_B \;|\psi\rangle|\beta_{00}\rangle \\
& + |\beta_{11}\rangle\langle\beta_{11}|_{A_1A_2}\otimes I_B \;|\psi\rangle|\beta_{00}\rangle
\end{aligned}$$

$$\begin{aligned}
= \;& \frac{1}{2}|\beta_{00}\rangle(a|0\rangle + b|1\rangle) \quad * \\
& + \frac{1}{2}|\beta_{01}\rangle(a|1\rangle + b|0\rangle) \\
& + \frac{1}{2}|\beta_{10}\rangle(a|0\rangle - b|1\rangle) \\
& + \frac{1}{2}|\beta_{11}\rangle(a|1\rangle - b|0\rangle)
\end{aligned}$$

example of calculation (line $*$)

$$\boxed{\begin{aligned}\langle\beta_{00}|00\rangle &= \tfrac{1}{\sqrt{2}} \\ \langle\beta_{00}|10\rangle &= 0 \\ \langle\beta_{00}|01\rangle &= 0 \\ \langle\beta_{00}|11\rangle &= \tfrac{1}{\sqrt{2}}\end{aligned}}$$

$$|\beta_{00}\rangle\langle\beta_{00}|\otimes I\frac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\Big[\langle\beta_{00}|00\rangle\, a|\beta_{00}\rangle|0\rangle + \langle\beta_{00}|01\rangle\, a|\beta_{00}\rangle|1\rangle$$
$$+ \langle\beta_{00}|10\rangle\, b|\beta_{00}\rangle|0\rangle + \langle\beta_{00}|11\rangle\, b|\beta_{00}\rangle|1\rangle\Big]$$
$$= \frac{1}{\sqrt{2}}\left[\frac{a}{\sqrt{2}}|\beta_{00}\rangle|0\rangle + \frac{b}{\sqrt{2}}|\beta_{00}\rangle|1\rangle\right] = \frac{1}{2}|\beta_{00}\rangle\underbrace{(a|0\rangle + b|1\rangle)}_{\text{properly normalized.}}$$

→ All 4 outcomes occur with equal probability $\left(\frac{1}{4}\right)$