CS333 - Problem Set 8

Due: Wednesday, April 25th before class

See final page for hints.

1. Quantum Fourier Transform and Period Finding. For a standard basis state $|x\rangle \in \mathbb{C}^t$,

$$QFT_t|x\rangle = \frac{1}{\sqrt{t}} \sum_{y=0}^{t-1} e^{2\pi i xy/t}|y\rangle, \qquad QFT_t^{-1}|x\rangle = \frac{1}{\sqrt{t}} \sum_{y=0}^{t-1} e^{-2\pi i xy/t}|y\rangle \tag{1}$$

   (a) **[6 points]** Show that

$$QFT_t|0\rangle = \frac{1}{\sqrt{t}} \sum_{y=0}^{t-1} |y\rangle. \tag{2}$$

   (b) **[6 points]** Show that QFT_t^{-1} really is the inverse of QFT. In other words, show:

$$QFT_t^{-1} QFT_t = I. \tag{3}$$

   (c) **[3 points]** Given a function f, let $P_k(f(x)) = f(x+k)$. What is a connection between P_k and period finding? (There is nothing quantum in this problem.)

   (d) **[6 points]** Let P denote the unitary operation that adds 1 modulo t. In other words, for any $x \in \{0, 1, \ldots, t-1\}$, $P|x\rangle = |x+1 \bmod t\rangle$. Show that the states that result from applying QFT_t to standard basis states are eigenvectors of P. That is, show

$$P \frac{1}{\sqrt{t}} \sum_{y=0}^{t-1} e^{2\pi i xy/t}|y\rangle = \lambda_y \frac{1}{\sqrt{t}} \sum_{y=0}^{t-1} e^{2\pi i xy/t}|y\rangle \tag{4}$$

   where λ_y is a complex number. What is λ_y? (This problem is meant to show you that there is some relationship between periodic functions and Fourier states.)

2. Let p be a prime number. Suppose you are given a black-box function $f\colon \{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, p-1\} \to \{0, 1, \ldots, p-1\}$ such that $f(x, y) = f(x', y')$ if and only if $y' - y = m(x' - x) \bmod p$ for some unknown integer m. In other words $f(x, mx + b) = C_b$, where C_b is a constant that depends on b. This means that for all points on the line $y = mx + b$, f has the same value. However, for different values of b, the function takes different values. Your goal is to determine $m \bmod p$ using as few queries as possible to f, which is given by a unitary operation U_f satisfying $U_f|x\rangle_A|y\rangle_B|z\rangle_C = |x\rangle_A|y\rangle_B|(z + f(x, y)) \bmod p\rangle_C$ for all $x, y, z \in \{0, 1, \ldots, p-1\}$. (Note that each of the three registers is a p-dimensional state.)

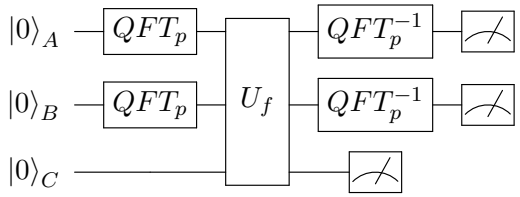
(a) **[3 points]** Consider the case that $p = 3$. Here is a truth table for a function of the above form. What is m?

x	y	$f(x,y)$
0	0	1
0	1	2
0	2	0
1	0	2
1	1	0
1	2	1
2	0	0
2	1	1
2	2	2

(5)

(b) **[6 points]** What is the classical query complexity of this problem?

(c) Consider the following circuit (which is very similar to the period finding circuit!!)



(6)

i. **[3 points]** What is the state of the system after the first time step (the two parallel QFTs)?

ii. **[6 points]** Show that the state after applying U_f is

$$\frac{1}{\sqrt{p}} \sum_{b=0}^{p-1} \left(\frac{1}{\sqrt{p}} \sum_{x=0}^{p-1} |x, mx + b\rangle_{AB} \right) |f(0,b)\rangle_C. \tag{7}$$

iii. **[3 points]** Argue that when we measure register C, each outcome occurs with equal probability. If the outcome is $|f(0,b^*)\rangle$, what is the state of the system after measurement?

iv. **[6 points]** If the outcome on register C is $|f(0,b^*)\rangle$, show that the final state after the last two inverse QFTs is

$$\frac{1}{\sqrt{p}^3} \sum_{j,l=0}^{p-1} e^{-2\pi i l b^*/p} \left(\sum_{x=0}^{p-1} e^{-2\pi i x(j+lm)/p} \right) |j\rangle|l\rangle \tag{8}$$

v. **[6 points]** Explain why when you measure the remaining state in the standard basis, you will get an outcome $|l\rangle|j\rangle$ where $j \equiv -lm \mod p$.

vi. **[6 points]** It is a fact from number theory that every number except 0 has a multiplicative inverse $\mod p$. In otherwords, if $j \not\equiv 0 \mod p$, there exists j^{-1} such that $jj^{-1} \equiv 1 \mod p$. Use this fact to explain how to learn m from the outcome of the final measurement.

Hints!

- $1b$: There are several ways to do this, but one way it to show that the operation takes every standard basis state to itself.
- $2ci$: Since $y = mx + b$, we can replace the variable y with the expression $mx + b$. Then $f(x, mx + b) = f(0, b)$ for all x because of the promise.