CS333 - Problem Set 5

Due: Wednesday, April 4th before class

See final page for hints.

1. Consider the state

$$|\phi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle. \tag{1}$$

   (a) **[6 points]** Suppose the first qubit of $|\phi\rangle$ is measured in the standard basis, i.e. $\{|0\rangle, |1\rangle\}$. What is the probability of obtaining outcome $|0\rangle$, and in the event that this outcome occurs, what is the resulting state of the second qubit?

   (b) **[6 points]** Suppose the second qubit of $|\phi\rangle$ is measured in the basis $\{|+\rangle, |-\rangle\}$. What is the probability of obtaining $|+\rangle$, and in the event that this outcome occurs, what is the resulting state of the first qubit?

   (c) **[6 points]** Suppose the first qubit of $|\phi\rangle$ is measured in the standard basis and the second qubit is measured in the $\{|+\rangle, |-\rangle\}$ basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

   (d) **[6 points]** **This is a big picture question** What did you learn from this problem?

2. Alice and Bob would like to share the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Unfortunately, they do not initially share any entanglement. But fortunately, they have a mutual friend, Charlie, who shares a copy of $|\beta_{00}\rangle$ with Alice and another copy of $|\beta_{00}\rangle$ with Bob.

   (a) **[6 points]** Write the initial state using ket notation as a superposition of standard basis states, and use subscripts to indicate which qubit is in which person's posession. The first qubit should belong to Alice, the second and third qubits belong to Charlie (the second is entangled with Alice's qubit and the third is entangled with Bob's qubit), and the fourth qubit belongs to Bob.

   (b) **[6 points]** Suppose Charlie performs a Bell measurement on his two qubits (one of which is entangled with Alice and the other of which is entangled with Bob). For each possible measurement outcome, give the probability with which it occurs and the resulting post-measurement state for Alice and Bob.

   (c) **[6 points]** Consider the system after Charlie's measurement, and design a protocol whereby Charlie sends a classical message to Alice, and Alice applies a unitary to her quantum state based on that message, such that after doing this, Alice and Bob share the state $|\beta_{00}\rangle$.

3. The CCCNOT (triple-controlled NOT) gate is a four-bit reversible gate that flips its fourth bit if and only if the first three bits are all in the state 1.

(a) **[6 points]** Show how to compute CCCNOT using AND, OR, NOT and FANOUT gates.

(b) **[6 points]** Show how to implement a CCCNOT gate using TOFFOLI gates. You may use additional ancillae as needed. You may assume that ancilla bits start with a particular value, either 0 or 1, provided you return them to that value. For a challenge, give a circuit that works regardless of the values of the ancilla initially.

(c) **[6 points]** Show that a Toffoli gate cannot be implemented using any number of CNOT gates, (if you can ONLY use CNOT gates) with any amount of workspace.

(d) **[3 points]** **This is a big picture question** What does the previous problem tell you about CNOT vs TOFFOLI.

4. In class, we discussed how classical probabilistic computation can be represented mathematically using vectors to represent the state of the system, and left stochastic matrices to represent operations. This method can also be used to describe reversible deterministic (non-probabilisitc) classical computation. If the system is n bits, and those bits are in the state $s \in \{0,1\}^n$ (i.e. s is some n bit string), then we can represent s using the vector $|s\rangle$ (where here, $|s\rangle$ doesn't represent a quantum state, but just represents the standard basis vector). In this reperesentation, deterministic gates are reversible matrices.

(a) **[3 points]** What class of matrices describe the set of possible gates in this model? (For example, in the case of quantum, we had *unitary*, in the case of probabilistic, we had *left stochastic.*)

(b) Please give a matrix representation of each of the following gates:

i. **[3 points]** NOT (acts on a single bit and flips the value)

ii. **[3 points]** CNOT (acts on two bits and flips the value of the second bit if the first has value 1)

iii. **[3 points]** TOFFOLI (acts on 3 bits and flips the value of the third bit if both of the first bits have value 1)

(c) **[6 points]** **This is a big picture question** Are all of the matrices from part (b) also valid left stochastic matrices? Are all of these matrices also valid unitaries? Based on your answers, please comment on the ability of probabilistic computers to simulate deterministic computers, and the ability of quantum computers to simulate determistic computers. If it is possible to do this simulation, explain how you would do it.

Hints!

1. (a) $Pr(|0\rangle) = 5/9$ (barring an error!)
   (b) $Pr(|+\rangle) = 13/18$ (barring an error!)
   (c) No hint
   (d) No hint
2. (a) No hint
   (b) No hint
   (c) Think superdense coding.
3. (a) No hint
   (b) Try using an extra ancilla bit that is initially in the state 0, and think about how you can store information about the values of two of the bits there temporarily. I'll leave it as a challenge to figure out the case when you can't set the value of the ancilla.
   (c) Think about all of the possible outputs of CNOT are. Can it do anything other than add mod 2? What are it's possible outputs given x, y, and all possible ancilla??
4. (a) If you are having trouble, try doing part b first. If you don't know the name for this type of matrix, just give a description of its properties.
   (b) Make sure your matrices are the correct dimensions. How big should a gate that acts on 2 bits be?
   (c) No hints