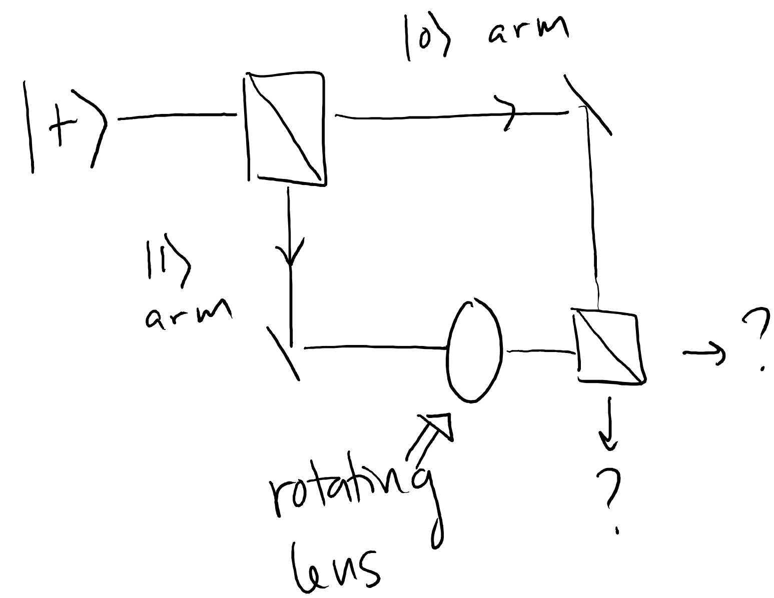
CS333 - Problem Set 3

Due: Wed, Mar 7

1. **[6 points each]**



Consider the interferometer in the figure. What will be the state of the photon that exits the interferometer in the following situations?

   (a) Lens acts as $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle \to \cos(\theta+\pi)|0\rangle + \sin(\theta+\pi)|1\rangle$.

   (b) Lens acts as $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle \to \cos(\theta+\pi/2)|0\rangle + \sin(\theta+\pi/2)|1\rangle$.

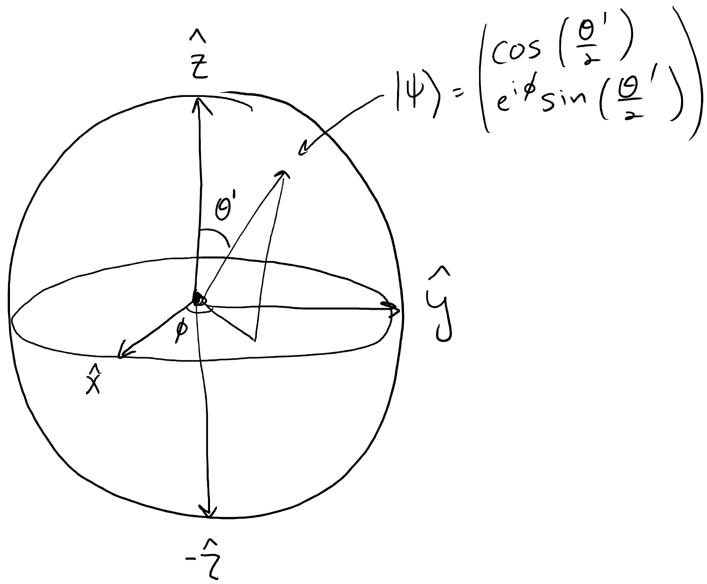
2. **[3 points]** In class, we discussed a rotating lens that corresponds to a unitary U_χ that acts in the following way: if a photon with polarization $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ enters the lens, it is transformed into a photon with polarization $\cos(\theta+\chi)|0\rangle + \sin(\theta+\chi)|1\rangle$. Please describe the unitary operation U_χ in matrix form. (Hint - if it's been a while since you've played with trigonometric functions, you should look up angle addition formulas for sin and cos.)

3. Consider the operation $I - 2|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a valid qubit state and I is the 2×2 identity matrix. (This unitary will be important for the quantum searching algorithm!)

   (a) **[6 points]** Show $I - 2|\psi\rangle\langle\psi|$ is a unitary operation.

   (b) **[6 points]** Describe what U does.

4. **[3 points each]**



In the last problem set, you learned that single qubit states can be represented as vectors on the surface of the sphere. Single qubit unitaries can be thought of as rotations of the sphere (the rotation rotates the vectors, and so changes the state.) A rotation is defined by two quantities: an axis of rotation, and an angle of rotation. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{1}$$

is a rotation around the $\hat{z}$ axis (because this unitary doesn't change the state $|0\rangle$). The direction of rotation is found using the "right hand rule:" point your right thumb in the direction of the rotation (in this case $\hat{z}$), and then the rotation is in the direction your fingers move when you start with an open hand and then close it. So in this case, because the unitary turns $|+\rangle$ into $|\rightarrow\rangle$, the rotation is $\pi/2$ (90°) of the Bloch sphere.

What is the axis of rotation and angle of rotation of the Bloch sphere corresponding to each of the following unitaries:

(a) I

(b) X

(c) Z

(d) Y

(e) H

5. Let $|\psi\rangle$ and $|\psi^{\perp}\rangle$ be two orthonormal basis states, and suppose you know that a single qubit unitary U takes

$$|\psi\rangle \rightarrow |\phi\rangle \tag{2}$$
$$|\psi^{\perp}\rangle \rightarrow |\chi\rangle. \tag{3}$$

(a) **[6 points]** Show $|\phi\rangle$ and $|\chi\rangle$ are orthnormal.

(b) **[6 points]** Describe the unitary using ket-bra notation (i.e. using terms of the form $|\cdot\rangle\langle\cdot|$), and then using this notation, show that $UU^\dagger = I$.

(c) **[3 points]** Is U the same as the unitary U', which takes

$$|\psi\rangle \to |\phi\rangle \tag{4}$$

$$|\psi^\perp\rangle \to -|\chi\rangle. \tag{5}$$

(d) **[3 points]** Is U the same as the unitary U'', which takes

$$|\psi\rangle \to -|\phi\rangle \tag{6}$$

$$|\psi^\perp\rangle \to -|\chi\rangle. \tag{7}$$

(e) **[6 points]** What did you learn about unitaries from this problem?

6. **[Optional - if you are not familiar with Taylor approximations, or want extra practice]**. When you have a function $f(x)$, you can write $f(x)$ in terms of a sum of powers of x, based on the derivatives of f relative to x. You don't need to know the details of Taylor series expansions for this class, but if you are curious, there is lots more info online. When x is very small (much less than 1), large powers of x don't affect the value of the function, and we can approximate $f(x)$ by the lowest power(s) of x in the sum. For this class, the relevant approximations that you should know are:

   - $(1+x)^n = 1 + nx + O(x^2)$
   - $\cos(x) = 1 - \frac{x^2}{2} + O(x^4)$
   - $\sin(x) = x + O(x^3)$
   - $e^{ax} = 1 + ax + \frac{(ax)^2}{2} + O(x^3)$

   For each of the following, give an apporximation to lower order in x.

   (a) $\frac{1}{1-x}$

   (b) $\tan(x)$

   (c) $\frac{1}{e^{ix}+e^{-ix}}$

7. How long did you spend on this homework?