Deutsch's algorithm analyzes a 1-bit function f:

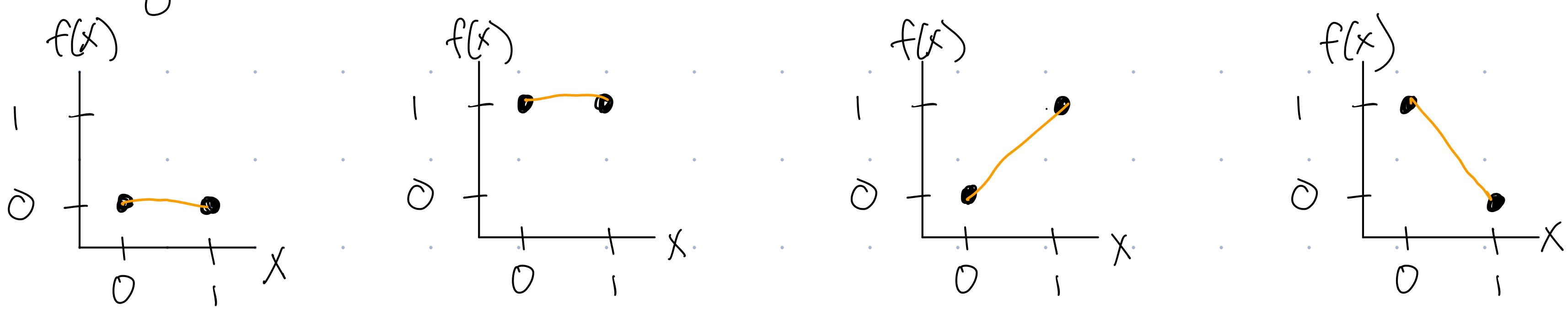
x	$f(x)$
0	$f(0)$
1	$f(1)$

$f(0), f(1) \in \{0,1\}$

ex: $f(x) = \begin{cases} 1 & \text{if it will rain at time } x \\ 0 & \text{if it will not rain} \end{cases}$

$x = \begin{cases} 0 \to \text{day} \\ 1 \to \text{night} \end{cases}$

Only 4 1-bit Functions:



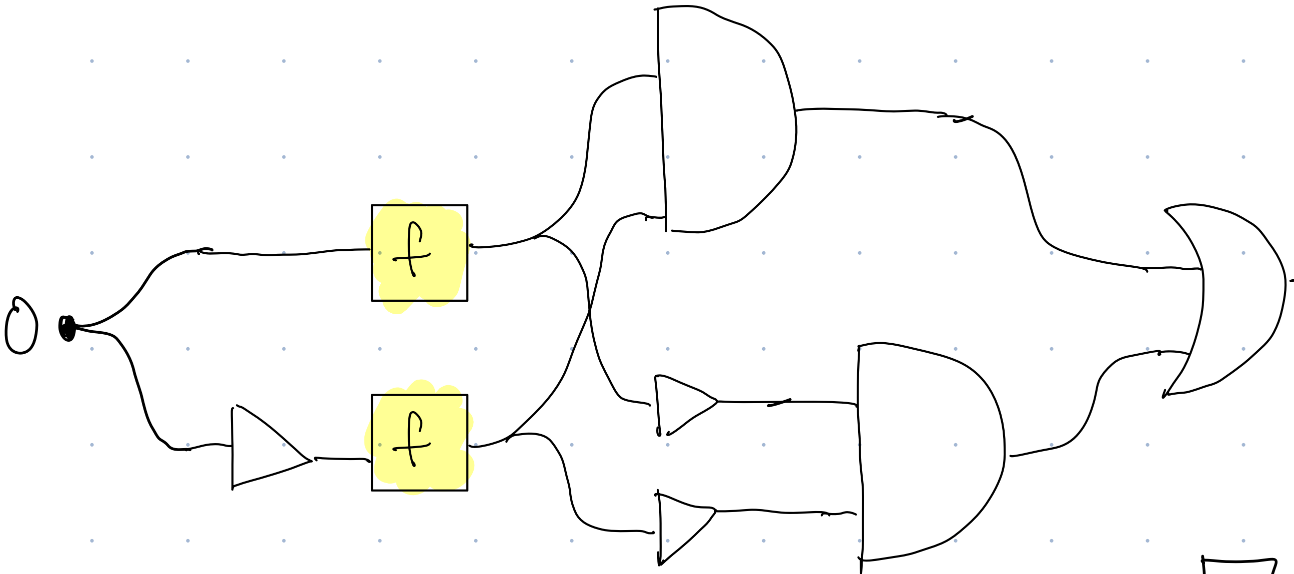
flat (even)

sloped (balanced)

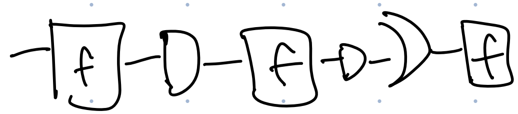
Deutsch's Problem: Given query access to f, determine if f is flat or sloped

A Classical Algorithm (circuit diagram)

- time goes L → R
- Wires are bits
- Shapes are gates



1 ↔ flat
0 ↔ sloped



Query Complexity = 2   **Time Complexity** = 5, 8, time complexity of f?

What is the minimum number of classical queries to f needed to determine flat/sloped

A) 0

B) 1

$\boxed{\text{C) 2}}$

Deutsch's Problem

- Classical Query Complexity: 2
- Quantum Query Complexity: 1

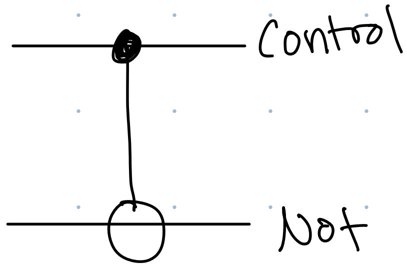
Why do we care?!
- Provable quantum computing advantage
- Techniques

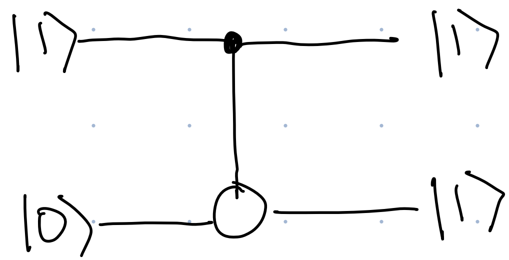
- Might be useful if f is really hard to compute

Quantum Gates (Circuit picture)

- time L → R
- wires are qubits
- gates + measurements are shapes

$|0\rangle$  Initialize qubit in $|0\rangle$

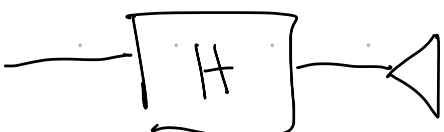




Measure in standard basis $M = \{|0\rangle, |1\rangle\}$

Measure in Hadamard

$|+\rangle \longrightarrow |0\rangle$



$\approx \quad M = \{|+\rangle, |-\rangle\}$

$|-\rangle \longrightarrow |1\rangle$

Need gate for f. What about:

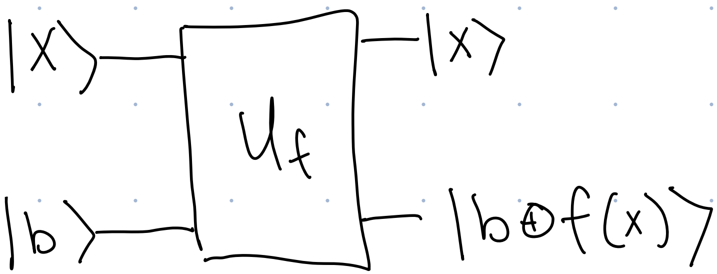
$|0\rangle \to |f(0)\rangle$ $\quad$ $|0\rangle \to |0\rangle$ $\quad$ $f(0) = f(1) = 0$

$|1\rangle \to |f(1)\rangle$ $\quad$ $|1\rangle \to |0\rangle$

Explain why this is not an allowed gate.

(State → State, Reversible)

Instead:



$|x\rangle|b\rangle \longrightarrow |x\rangle|b \oplus f(x)\rangle$

To interpret, $x, b \in \{0,1\}$

(standard basis states)

Interpretation:

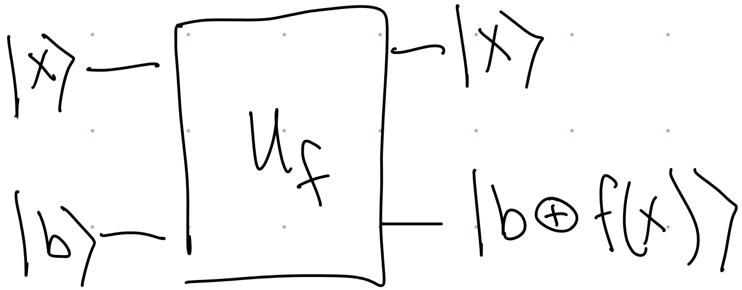
$|0\rangle|0\rangle \to |0\rangle|0 + f(0) \bmod 2\rangle$

$|0\rangle|1\rangle \to |0\rangle|1 + f(0) \bmod 2\rangle$

$|1\rangle|0\rangle \to |1\rangle|0 + f(1) \bmod 2\rangle$

$|1\rangle|1\rangle \to |1\rangle|1 + f(1) \bmod 2\rangle$

$U_f U_f = I$

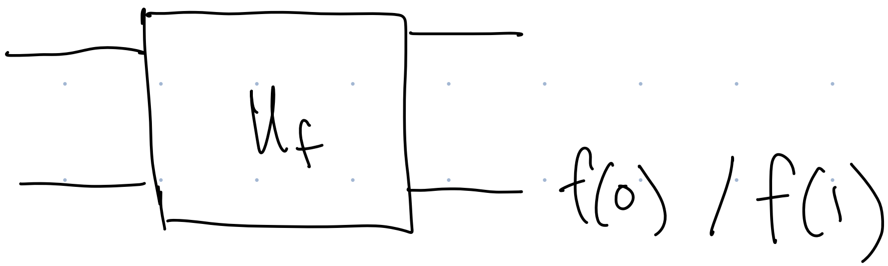


If $x = 0$ get info about $f(0)$

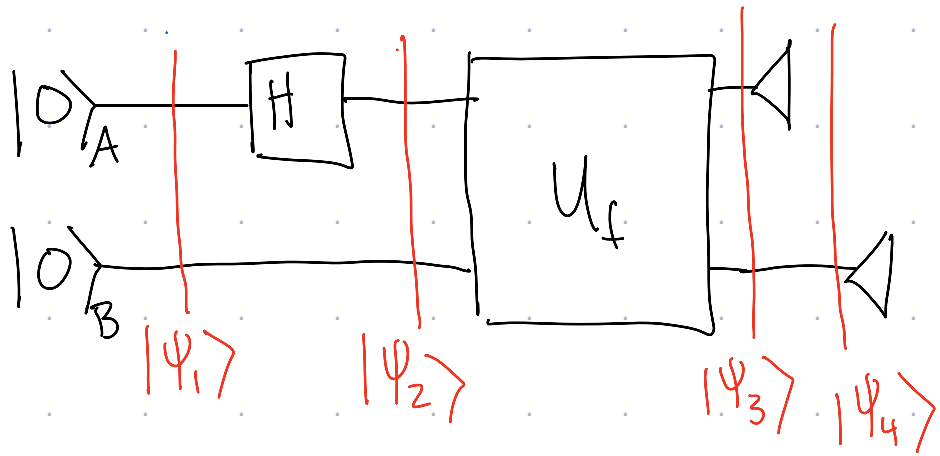
If $x = 1$ get info about $f(1)$

Idea – Superposition!

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



Analyzing a Quantum Circuit



U_f:

$$\underbrace{|x\rangle|b\rangle \to |x\rangle|b \oplus f(x)\rangle}$$

ONLY WORK IF $|x\rangle$, $|b\rangle$ are standard basis states!!!

$|\psi_1\rangle = |0\rangle_A \otimes |0\rangle_B = |0\rangle_A |0\rangle_B$ (first qubit is top qubit)

$|\psi_2\rangle = H \otimes I \, |0\rangle|0\rangle = (H|0\rangle) \otimes I|0\rangle = |+\rangle|0\rangle$

$|\psi_3\rangle = U_f |+\rangle|0\rangle \cancel{= |+\rangle|0 \oplus f(+)\rangle}$

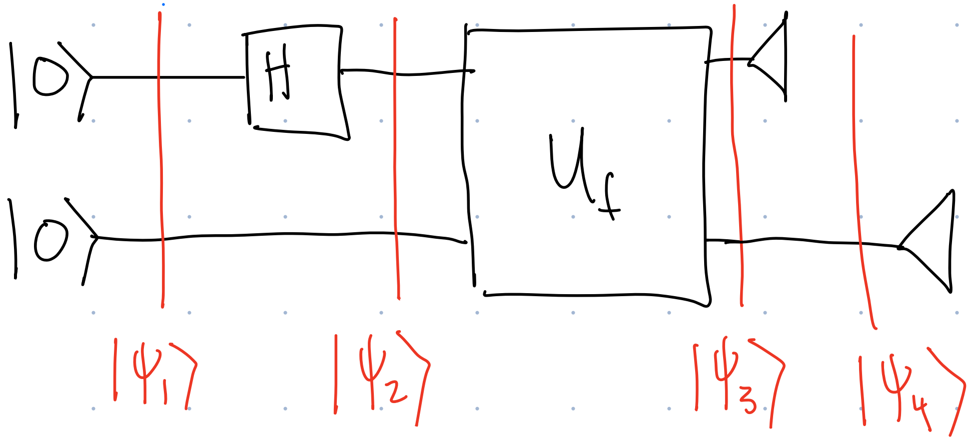
$(x=1 \quad b=0)$

$= U_f \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}} U_f |0\rangle|0\rangle + \frac{1}{\sqrt{2}} U_f |1\rangle|0\rangle$

$(|1\rangle|0 \oplus f(1)\rangle)$

$= \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle$

$=$



$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle$$

Partial Measurement:

Outcome $|0\rangle$

- Prob $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- Collapse : $|0\rangle|f(0)\rangle \longrightarrow |f(0)\rangle$

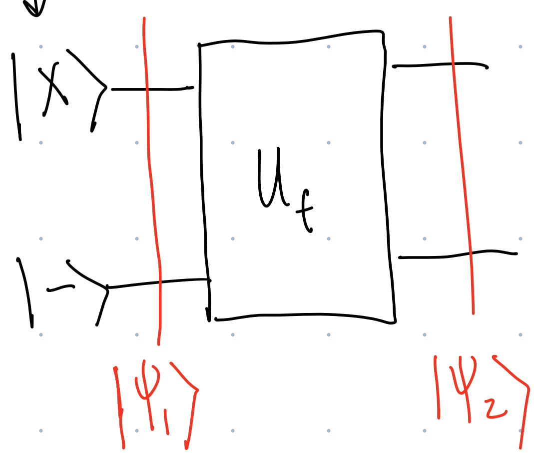
Outcome $|1\rangle$

- Prob $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- Collapse $|1\rangle|f(1)\rangle \longrightarrow |f(1)\rangle$

Mini-Hype Lesson

Group Exercise:

$x = 0$ or 1



$$|x\rangle|b\rangle \to |x\rangle|b \oplus f(x)\rangle$$

Show that $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|-\rangle$

Analyze outcome of this circuit: