# Dijkstra's Algorithm

Lecture 6

## Shortest Path Problem

Input: $G=(V,E)$, $w: E \to \mathbb{R}$, $s \in V$

Output: Shortest path from $s$ to $v$ $\forall$ $v \in V$

$\frac{1}{2}$ sequence matching for

| Contexts | Algorithm | Runtime |
|---|---|---|
| Acyclic graphs | Topo-sort + DP | $O(m)$ |
| no negative edges | Dijkstra's | $O(m \log n)$ |
| all edges have weight 1 | Breadth First Search | $O(m+n)$ |

**Goals**

- Describe Dijkstra's Algorithm
- Analyze runtime of Dijkstra's Algorithm
- Prove correctness (maybe)

**Supplementary**

- Bellman-Ford
- Official Hours Tues/Wed 10 noon

## Dijkstra's Algorithm

$X = \{s\}$ // set of visited vertices

$A[v] = \infty$ // length of shortest path from $s$ to $v$

$B[v] = \emptyset$ // shortest path

while there are edges

- Let $(v,w) \in E$ with $v \in X$, $w \notin X$
- Let $(v^*, w^*) = \arg\min_{(v,w) \in E, v \in X, w \notin X} A[v] + w_{vw}$
- Add $w^*$ to $X$
- $A[w^*] = A[v^*] + w_{v^*w^*}$
- $B[w^*] = B[v^*] + (v^*, w^*)$

$\Rightarrow X = V$ & $B[v] \neq \emptyset$

Loop 1

$A[v] = 0$, $A[w] = 1$, $A[v^*] = \infty$, $A[w^*] = \infty$

$B[v] = \emptyset$, $B[w] = \begin{bmatrix} s \\ v \end{bmatrix}$, $B[v^*] = \emptyset$, $B[w^*] = \emptyset$

Loop 2

$A[v] = 0$, $A[w] = 1$, $A[v^*] = 2$, $A[w^*] = 4$

$B[v] = \emptyset$, $B[w] = \begin{bmatrix} s \\ v \end{bmatrix}$, $B[v^*] = \begin{bmatrix} s \\ v \\ v^* \end{bmatrix}$, $B[w^*] = \begin{bmatrix} s \\ v \\ w \\ w^* \end{bmatrix}$

## Correctness Proof

Claim: Dijkstra's algorithm is correct. (Specifically that Dijkstra's algorithm does not always work when there are negative edge weights. (Bad example is coming in 6.2 algorithm. Bonus: 6.2 has shortest shortest path.)

$A[v] = -2$

## Analyzing Runtime

What do we need efficiently do: select data structure work big at

Priority Queue / Heap

What do we need...

- Edges → Deleted
- Vertices → Deleted

Operations on Heap: $O(\log n)$

$\boxed{\text{If } w_{v^*w^*} < A[w^*] \text{ then } A[w^*] = A[v^*] + w_{v^*w^*}}$

$\boxed{\text{Decrease-Key}}$

Alternative: Big

$\boxed{\text{If } w_{v^*w^*} < A[w^*] \text{ then } A[w^*] = A[v^*] + w_{v^*w^*}}$

## Dijkstra's Algorithm

$X = \{s\}$, $A[s] = 0$ // visited vertices

$A[v] = \infty$ $\forall v \neq s$ // shortest distance from s

$B[v] = \emptyset$

while $|X| < n$

for vertices $v \in X$

for edges $(v,w) \in E$

if $w \notin X$

$A[w] = \min(A[w], A[v] + w_{vw})$

$O(m)$

$O(m)$

$O(m)