CS302 - Problem Set 9

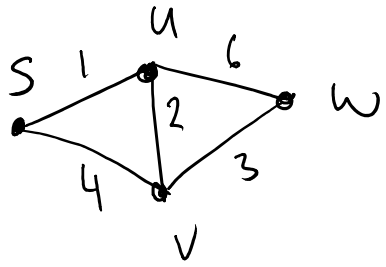
Due: Monday, Nov 20.

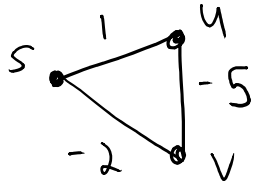
1. (**(*)) **[11 points]** Prove the BFS shortest path algorithm I described in class is correct. That is, prove that if the shortest distance from s to v is n, then $L[v] = n$ (and if s and v are not connected, then $L[v] = \infty$.)

   (Hints: Your proof will likely involve showing that closer nodes are explored first. Treat connected and disconnected cases separately.)

2. (*(*)) **[6 points]** What is the runtime of the BFS shortest path algorithm I described in class if the graph is given as an adjacency matrix instead of an adjacency list? (Let n be the total number of vertices in the graph, m the total number of edges, n_s the number of vertices connected to s, the starting nodes, and m_s the number of edges connected to s, the starting node.)

3. **[6 points each]** (*) Consider the following graphs and how Dijkstra's algorithm behaves on these instances. For each graph, for each loop of Dijkstra's algorithm, write the vertices in X at the beginning of the loop, the value of $A[v]$ for the vertex v most recently added to X, and calculate Dijkstra's criterion for each edge from X to $V - X$. Assume that s is the starting node in both graphs. For example, for the graph on the left, after intialization we have $X = \{s\}$, $A[s] = 0$, and the edges from X to $V - X$ are (s, u) with criterion value 1, and (s, v) with criterion value 4. Does the algorithm find the shortest path for each vertex in each graph?





4. Given a path from s to t in a graph $G = (E, V)$, we define the bottleneck of the path to be the largest weight of any edge on the path.

   (a) **[9 points]** (**(*)) Describe (using pseudocode) a modification of Dijkstra's algorithm that solves this problem. (Hint - the algorithm stays the same, just the criterion changes.)

   (b) **[11 points]** (**) Prove your algorithm finds the path with smallest bottleneck from s to every other vertex in G.

5. **[6 points each]** (*(*)) For the following statements, either explain why it is true (proof not required), or provide a counter example.

(a) Consider a graph G that is directed, has negative edge weights, but no negative cycles (a negative cycle is a cycle where the sum of edge-weights in the cycle have negative value.) Then there will always be a vertex where the incorrect distance is calculated.

(b) Consider a graph G that is directed, and that has a negative cycle that is reachable from s. Then there will always be a vertex where the incorrect distance is calculated.