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3. Divide & Conquer Multiplication (w/o Gauss's Trick)

$a = 4$ (4 recursive calls)

$b = 2$ (new problems ½ size)

$d = 1$ (constant number of n-bit additions)

$T(1)$ = constant

$b^d = 2^1 = 2 < 4 = a \Rightarrow$ Case 3

$T(n) = O(n^{\log_b a}) = O(n^{\log_2 4}) = O(n^2)$

4. Karatsuba Multiplication

$a = 3$ (3 recursive calls)

$b = 2$ (new problems ½ size)

$d = 1$ (constant # of n-bit additions)

$T(1)$ = constant

$b^d = 2^1 = 2 < 3 = a \Rightarrow$ Case 3

$T(n) = O(n^{\log_b a}) = O(n^{\log_2 3}) = O(n^{1.59})$

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5. Closest Pair

$a = 2$ (2 recursive calls)

$b = 2$ (new problems ½ size)

$d = 1$ (creating new matrices / going through 1/5)

$T(3), T(4) \leq$ constant

$b^d = 2^1 = 2 = a \Rightarrow$ case 1

$T(n) = O(n^d \log n) = O(n \log n)$

☆ a is the number of recursive calls at that level. Don't count recursive calls that happen in other recursive calls.

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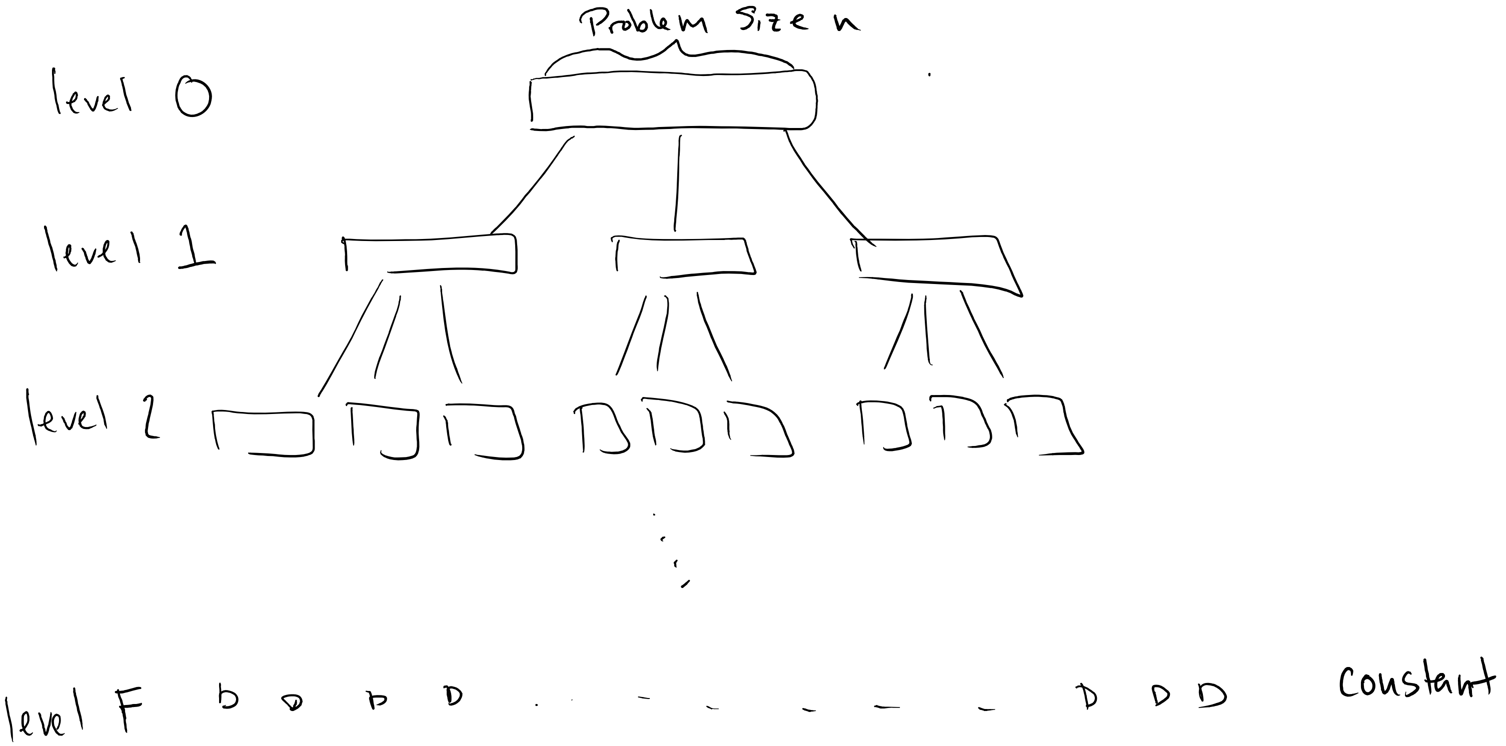
Proof of Master Method



Q. What is F (in terms of a, b, d)?

A) $\Theta(\log_b n)$ B) $\Theta(\log_d n)$ C) $\Theta(n^{\log_b d})$ D) $\Theta(b^{\log_d n})$

$\Uparrow$

Because at each level, problem size is divided by b. $\log_b n$ is number of times n can be divided by b before reaching a constant.

$$\text{constant}\cdot\underbrace{b\cdot b\cdots b}_{F} = n$$

$$c\,b^F = n$$

$$b^F = \frac{n}{c} \quad \text{take } \log_b \text{ of both sides} \rightarrow F\overbrace{\log_b b}^{1} = \log_b n - \log_b c$$

$$F = \log_b n - \log_b c \qquad F = \log_b n + \text{constant}$$

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Q. What is the total work done at level k (outside of recursive calls & in terms of a, b, d)?

operations

- $\boxed{a^k}$ subproblems at level k.
- level k subproblem size: $\boxed{\frac{n}{b^k}}$
- Work outside of recursive call required to solve $\underline{1}$ subproblem : $\boxed{\left(\frac{n}{b^k}\right)^d}$

$\Rightarrow$ Total work $\quad a^k\left(\frac{n}{b^k}\right)^d = \boxed{\left(\frac{a}{b^d}\right)^k n^d}$

Now we add up work done at all levels:

$$\sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k n^d$$

$$= n^d\left[\sum_{k=0}^{\log_b d} \left(\frac{a}{b^d}\right)^k\right]$$

Mutliplicative Distributive property

Geometric Series:

$$\sum_{k=0}^{F} r^k = \begin{cases} F+1 & \text{if } r = 1 \\ \dfrac{1-r^{F+1}}{1-r} & \text{otherwise} \end{cases}$$

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$$n^d\left[\sum_{k=0}^{\log_b n}\left(\frac{a}{b^d}\right)^k\right] = \begin{cases} a=b^d \rightarrow n^d(\log_b n+1) \rightarrow O(n^d\log_b n) \\ a>b^d \rightarrow n^d\left(\dfrac{1-\left(\frac{a}{b^d}\right)^{\log_b n+1}}{\boxed{1-\left(\frac{a}{b^d}\right)}}\right) \rightarrow O(n^d) \\ \text{or} \\ a<b^d \rightarrow \quad \rightarrow O(n^{\log_b a}) \end{cases}$$

constant

If $a<b^d$, then $\frac{a}{b^d}<1$, so $1-\left(\frac{a}{b^d}\right) \leq 1-\left(\frac{a}{b^d}\right)^{\log_b n+1} < 1$

so $\dfrac{1-\left(\frac{a}{b^d}\right)^{\log_b n+1}}{1-\frac{a}{b^d}} \leq$ constant

If $a>b^d$ then $\frac{a}{b^d}>1$ so $\dfrac{1-\left(\frac{a}{b^d}\right)^{\log_b n+1}}{1-\frac{a}{b^d}} = \dfrac{\left(\frac{a}{b^d}\right)^{\log_b n+1}-1}{\frac{a}{b^d}-1}$

$$\geq \frac{\left(\frac{a}{b^d}\right)^{\log_b n+1}}{\boxed{\frac{a}{b^d}-1}}$$

constant

$$\left(\frac{a}{b^d}\right)^{\log_b n+1} = \frac{a^{\log_b n+1}}{b^{(\log_b n+1)d}} = \frac{a^{\log_b n+1}}{(bn)^d}$$

n^d cancels with

$$a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}$$