Name ____________________________

CS200 - Functions Worksheet

1. Let $f : \{1,2,3\} \to \{1,2,3\}$. Which of the following is an incorrect representation of f?

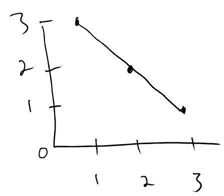
A

$1 \to 3$
$2 \to 2$
$3 \to 1$

B

$f(x) = -x + 4$

C

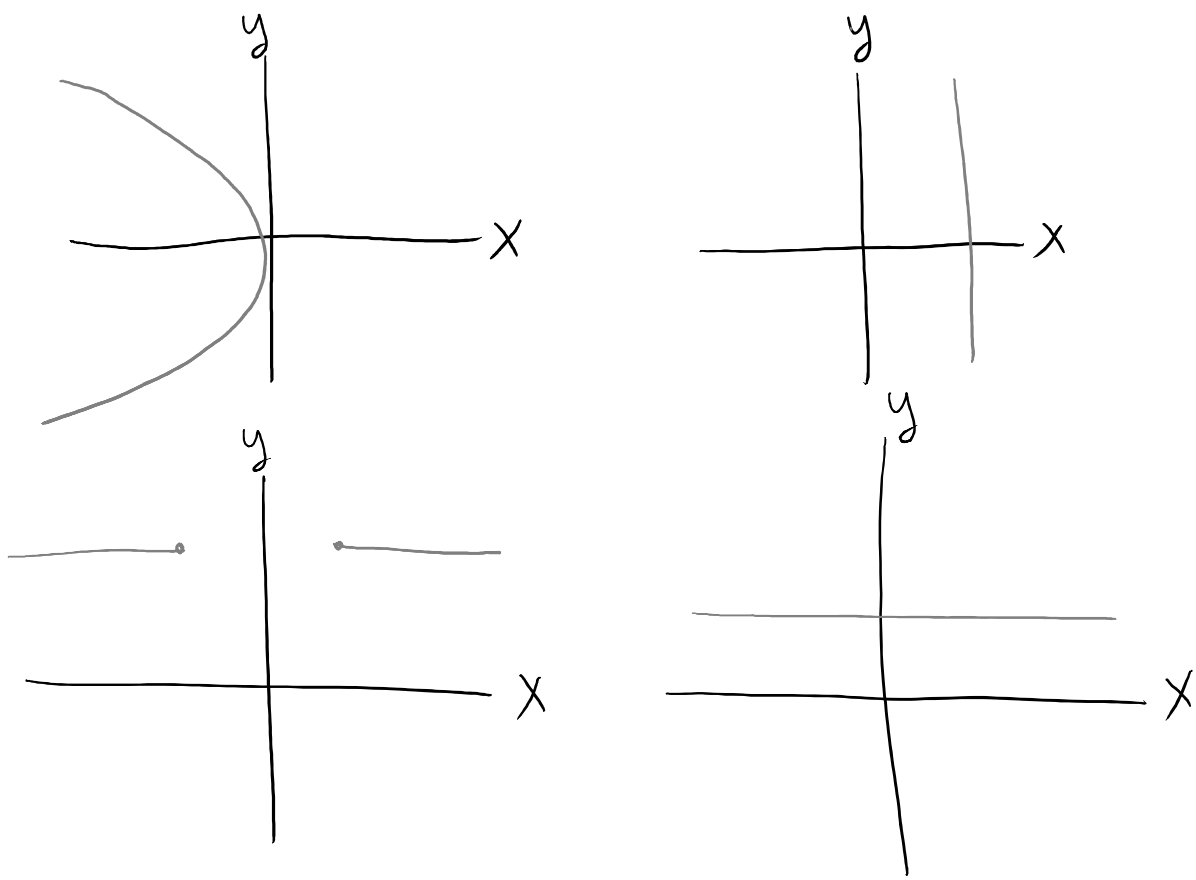


D

x	1	2	3
$f(x)$	3	2	1

Solution C - because the domain includes real numbers, not just $\{1,2,3\}$

2. Consider a function $f : \mathbb{R} \to \mathbb{R}$. If $y = f(x)$, we can depict f as a graph. Which of the following graphs (if any) could depict f?



Solution Only the lower right. The top two have a single input going to multiple different outputs. The lower left's domain is not $\mathbb{R}$.

3. Think of a real world function that is

   - surjective
   - injective
   - bijective
   - neither surjective nor injective

   Solution (Yours will differ!)

   - Surjective: The function from students in this class to months of the year, where the image of the student is their birth month.
   - Injective: The function from students in this class to middlebury e-mail addresses, where the image of the student is their e-mail.
   - Bijective: The function that takes the set of single dorm rooms in Coffin to the set of students who live in singles in Coffrin.
   - Neither (very likely): The function from the set of students in both sections of 200 to the days of the year, where the image of the student is their birthday.

4. When you write a function in python or a method in java, what are typical domains and co-domains?

   Solution Integers, Doubles, Strings, Booleans, etc.

5. English to Math, and just plain English: explain in words (using the new vocabulary you've learned) what each of the following means, and then express each using only mathematical notation.

   (a) A function $f : S \to G$ is surjective $\equiv$

   (b) A function $f : S \to G$ is injective $\equiv$

   Solution

   (a) A function is surjective if every element of the codomain has a preimage. In other words: $\forall x \in G, \exists y \in S : f(y) = x$.

   (b) A function is injective if no two elements of the domain map to the same element of the codomain. In other words: $\neg\exists a, b \in S : (a \neq b) \wedge (f(a) = f(b))$

6. A function is strictly increasing if $f(x) < f(y)$ whenever $x < y$. A function is increasing if $f(x) \leq f(y)$ whenever $x < y$.

   (a) Prove that a if $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then it is injective.

   (b) Prove that there exists an increasing function that is not injective.

Solution

(a) We prove the contrapositive. If f is not injective, then there exists $a, b \in \mathbb{R}$ where $a \neq b$ such that $f(a) = f(b)$. Without loss of generality, let $a < b$ (the proof will be the same when we set $b < a$). Then for f to be strictly increasing, we need $f(a) < f(b)$. Thus the function is not strictly increasing.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function that rounds each number up to the nearest multiple of 10. This function is increasing but not strictly increasing.

7. Let $f : \mathbb{Z} \to \mathbb{N}$ be the function $f(x) = x^2$. For each of the following, please give an explanation.

   (a) [**6 points**] Is f surjective?

   (b) [**6 points**] Is f injective?

Solution

(a) f is not surjective because, the square root of 5 is not an integer.

(b) f is not injective, because both -1 and 1 get mapped to the same value (1).

8. A relation R from the set A to B is a subset of $A \times B$. That is, $R \subset A \times B$. Explain why we can think of every function as a relation. Prove true or prove false that every relation represents a function.

Solution If we have a function $f : A \to B$, we can represent this using the relation $R \subset A \times B$, where $(a, b) \in R$ if and only if $f(a) = b$. However, this correspondance doesn't always work the other way. For example, we could have a relation $Q \subset A \times B$ where $(a, b_1) \in Q$ and $(a, b_2) \in Q$. In this case, it is unclear what $f(a)$ is.