CS200 - Problem Set 2

Please read the sections of the syllabus on problem sets and honor code before starting this homework.

For the following questions, I expect you to figure out some set notation on your own. I've given you a table (below) of all the relevant "vocab words" with math definitions. (See also, e.g. DMOI Set Introduction for further examples.) Pay close attention to inputs and outputs. You will need to memorize these symbols.

symbol	"inputs"	"output"	meaning
$\{a, b, c, \ldots\}$	elements $a, b, c \ldots$	set	"set containing the elements $a, b, c, \ldots$ (roster notation)
$\{a : P(a)\}$	elements a, predicate P	set	"the set of elements a such that $P(a)$ is true" (set-builder notation)
$a \in A$	element a, set A	predicate / statement	"a is an element of the set A"
$a \notin A$	element a, set A	predicate / statement	"a is not an element of the set A"
$A \subseteq B$	sets A, B	predicate / statement	"A is a subset of B" $a \in A \to a \in B$
$A \subset B$	sets A, B	predicate / statement	"A is a proper subset of B" $(a \in A \to a \in B) \wedge A \neq B$
$A \cap B$	sets A, B	set	"the intersection of A and B" $A \cap B = \{a : a \in A \wedge a \in B\}$
$A \cup B$	sets A, B	set	"the union of A and B" $A \cup B = \{a : a \in A \vee a \in B\}$
$A - B$ (or $A \setminus B$)	sets A, B	set	"A minus B" $A - B = \{a : a \in A \wedge a \notin B\}$
$\bar{A}$	set A	set	"the complement of A" $\bar{A} = \{a : a \notin A\}$ (there is an assumed "universe" set U, so really we mean $\bar{A} = \{a : a \notin A \wedge a \in U\}$
$\|A\|$	set A	integer	"the size (cardinality) of A" number of elements in A

1. (a) Describe the following sets in roster notation (list the first few elements). If the set is also "famous" give its symbol.

   i. $A = \{2^x : x \in \mathbb{N}\}$

   ii. $B = \{x : x \text{ is even and } x \in \{1, 3, 5\}\}$

iii. $C = \{x : x \text{ is greater than zero and odd}\}$

(b) Write the following in set-builder notation using as concise and as mathematical notation as possible

i. $\{2, 4, 6, 8, 10, 12\}$

ii. $\{1, 3, 5, 7, 9, 11, \dots\}$

iii. $\{1, 4, 9, 16, 25, 36, \dots\} \cap \{2, 4, 6, 8, 10, \dots\}$

iv. $\{1, 4, 9, 16, 25, 36, \dots\} \cup \{2, 4, 6, 8, 10, \dots\}$

v. **Challenge** $\overline{\{1, 4, 9, 16, 25, 36, \dots\} \cup \{2, 4, 6, 8, 10, \dots\}}$ where the universe is $\mathbb{N}$,

(c) Let $A = \{1, 2\}$ and $B = \{1, 2, \{3, 4\}\}$

i. Is $A \subset B$?

ii. Is $A \subseteq B$?

iii. Is $A \subset A$?

iv. What is $A - B$?

v. What is $A \cup B$?

vi. What is $A \cap B$?

vii. If B is the universe, what is $\bar{A}$?

(d) Which of the following are the empty set:

i. $\{x : x \text{ is odd and } 7 < x < 9\}$

ii. $\{0\}$

iii. $\{\emptyset\}$

iv. $\mathbb{Z} \cap \mathbb{Q}$

2. You have to be careful about distributing the $\neg$ symbol. You can not just do it willy-nilly! Prove the following either using a truth table proof or using an English explanation.

(a) $\neg(p \to q) \not\equiv \neg p \to \neg q$ ($\not\equiv$ means not equivalent. Very important!! The $\neg$ does not distribute!! These expressions are not the same.)

(b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ DeMorgan's Rule (distibrution with $\wedge$ changed to $\vee$)

(c) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ DeMorgan's Rule (distibrution with $\vee$ changed to $\wedge$)

3. *Quantifiers*
Consider the following statement:

$$\forall x, \exists y : (y > x) \wedge (\forall z, ((z \neq y) \wedge (z > x)) \to (z > y))$$

(a) If the domain of x, y and z is $\mathbb{Z}$, state whether the statement is true or false. Justify your answer.

(b) If the domain of x, y and z is $\mathbb{R}$, state whether the statement is true or false. Justify your answer.

4. (a) Write the sentence "There exists an integer that is not divisible by 2 and not divisible by 3," using mathematical notation.

(b) Now use de Morgan's rules (twice) to write a sentence using math that is logically equivalent to your sentence from part (a) "There exists an integer that is not divisible by 2 and not divisible by 3," but that uses a different quantifier from what you previously used. (If you used $\exists$ above, you should use $\forall$ here, or vice versa.)

(c) Translate your sentence from part (b) back into English. Write it in as natural a way as possible.

5. Read section 5.3 of Book of Proof by Richard Hammack. Then read the following poorly written proof of the statement: $\forall n \in \mathbb{Z}$, if n is even, then n^2 is even.

Proof:

1. Let $n =$ an integer.
2. Suppose n is even.
3. Then $n = 2k$.
4. $n^2 = (2k)^2$, $(2k)^2 = 4k^2$, so $4k^2 = 2(2k^2)$
5. Since $(2k^2)$ is an integer, I've shown it is even.

(The sentences in the proof are numbered to make it easier to reference specific lines in your answer.)

Please identify sentences that violate Hammack's mathematical writing guidelines and explain why. (A sentence can violate multiple guidelines, and so can be included multiple times.)

6. Let S be a set of students in a class and $f(s)$ be the score obtained by student s in an exam. Translate the English description of each predicate or statement below into a logical formula using only mathematical notation. When writing a formula for a statement or a predicate, you may use any propositions/predicates that you have previously defined. For example, you can use $H(n)$ in your definition of $B(s)$. Use the $\equiv$ symbol in your answer. For example, for part a, you should write $H(n) \equiv \ldots$.

These are challenging! To start each one, think about whether it should start with a "for all," or "there exists" quantifier (or whether you can write it without quantifiers and instead as a direct expression of previous predicates.) Next think about what set the domain should be for that quantifier. Then keep going :)

Things to remember: Make sure you don't mix types: don't put a number as input to a function whose input should be a student or vice-versa, or put a $\wedge$ between numbers ($\wedge$ can only go between predicates or statements!) Keep track of which variables you quantify and which you don't. For example, in problem (b), since s is the input to the predicate B, s should not be quantified (see class notes.) Also remember that if, for example, you create variables $x, y \in \mathbb{Z}$, it could be that $x = y$ unless you specify otherwise.

Your answer should only use mathematical notation, and try to make as concise as possible.

(a) Predicate $H(n)$ asserts: n is the highest score that any student got on the exam.

(b) Predicate $B(s)$ asserts: student s got the highest score. (This one doesn't need a quantifier)

(c) Statement p asserts: at least two students got the highest score.

(d) Predicate $M(n)$ asserts: if any two students got the same score, that score is at least n.

(e) Predicate $R(s)$ asserts: student s got 10 points less than the highest score.

(f) Statement t asserts: the second highest score in the class is 10 points less than the highest score.

7. How long did you spend on this homework?