CS200 - Worksheet 3

1. Create a recurrence relation for the worst case runtime of the following algorithm for binary search when $f - s + 1 = n$. You may assume n is a power of 2. Use the iterative method to solve the recurrence relation.

Algorithm 1: `BinarySearch`(A, x, s, f)

Input : Sorted (in increasing order) array of integers A, an integer x that occurs in the array, a starting index s and an ending vertex f

Output: An index i such that $A[i] = x$.

if $s == f$ **then**
  return s;
end
$mid = \lfloor (s + f)/2 \rfloor$;
if $A[mid] < x$ **then**
  return `BinarySearch`$(A, x, mid + 1, f)$
else
  return `BinarySearch`(A, x, s, mid)
end

Solution Let $T(n)$ be the run time of the algorithm, when $f - s + 1 = n$. Then

$$T(n) = T(n/2) + C \tag{1}$$

where C is some constant. The initial conditions are

$$T(1) = B \tag{2}$$

where B is some constant.

Then using the iterative approach, we have

$$\begin{aligned} T(n) &= T(n/4) + C + C = T(n/4) + 2C \\ &= T(n/8) + C + 2C = T(n/8) + 3C. \end{aligned} \tag{3}$$

We see the pattern is

$$T(n) = T(n/2^k) + k \times C. \tag{4}$$

The base case is when $2^k = n$, which happens when $k = \log_2(n)$. Plugging this in, we have

$$T(n) = T(1) + \log_2(n) \times C = O(\log_2(n)). \tag{5}$$

2. Let $K(n)$ be the size of the set of n-digit numbers that have an even number of 0's. Create a recurrence relation for $K(n)$. What is $K(3)$? (Hint 0: remember zero is even. Hint 1: think about the possible options for the value of the final digit. of the number. Hint 2: The size of the set of numbers that *don't* have an even number of 0's is the total number of elements minus the set of numbers that *do* have an even number of 0's.)

   Solution The base case is $K(1) = 9$ since the only way you can have a 1-digit number with an even number of 0s is to have no 0s.

   For the recurrence relation, We can break up the set of n-digit numbers with an even number of 0s into those that have a 0 in the final position, and those that don't. For those that have a final 0, there must be an odd number of 0s in the $n-1$ digits preceding the final digit. The number of $(n-1)$-digit numbers with an odd number of 0s is the total number of $(n-1)$ digit numbers (which is 10^{n-1}), minus the number of $(n-1)$-digit numbers with an even number of 0s (which is $K(n-1)$). Thus the number of n-digit numbers with an even number of 0s and the last digit equal to 0 is

$$10^{n-1} - K(n-1). \tag{6}$$

   In the case that the final digit is not a zero, then for there to be an even number of 0s, we must have an even number of zeros in the first $n-1$ digits. There are $K(n-1)$ such numbers. Since we have a choice of 9 digits for the last digit, there are $9 \times K(n-1)$ numbers that don't have a final digit 0. Thus the number of n-digit numbers with an even number of 0s and the last digit equal to 0 is

$$9K(n-1). \tag{7}$$

   Putting it all together, we have

$$K(n) = 9K(n-1) + 10^{n-1} - K(n-1) = 8K(n-1) + 10^{n-1}. \tag{8}$$

   Thus

$$K(3) = 8K(2) + 10^2 = 8(8K(1) + 10^1) + 10^2 = 8 \times 8 \times 9 + 110 = 686. \tag{9}$$

3. Create a recurrence relation for the number of ways a person can climb n stairs if the person can take one stair or two stair at a time. How many ways can this person climb a flight of 8 stairs?

   Solution Let $S(n)$ be the number of ways the person can take n stairs. For the intitial conditions, we have

$$T(1) = 1 \tag{10}$$

   because there is only one way to take one stair, and

$$T(2) = 2 \tag{11}$$

because the person could either take the 2 stairs at once, or could go up one at a time.

Now for the recurrence relation, if the person takes n stairs, they could either take the last stair on it's own, or they could take the last two stairs together.

If they take the final stair as one step, there are $T(n-1)$ ways that they could take the first $n-1$ stairs to get there, so there are $T(n-1)$ ways of taking the final stair alone.

If they take the final two stairs together, there are $T(n-2)$ ways that they could have gotten to the $(n-2)$th stair, and then one way that they can take the final two together.

Thus

$$T(n) = T(n-1) + T(n-2). \tag{12}$$

So

$$\begin{aligned} T(8) &= T(7) + T(6) = T(6) + T(5) + T(5) + T(4) = T(6) + 2T(5) + T(4) \\ &= T(5) + T(4) + 2(T(4) + T(3)) + T(3) + 2 \\ &= T(5) + 3(T(4) + T(3)) + 2 = T(4) + T(3) + 4(T(3) + T(2) + T(2) + T(1)) + 2 = \end{aligned} \tag{13}$$

...I got bored, but you see where this is going, hopefully!