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What did we learn about probability last time?

- Conditional Probability
- Independent Random Variables.

<u>Today</u> (final new topic in counting/probability)

Random Variables ⟵ Total misnomer : Not random
Not variable

<u>def</u>: Given a sample space S, a random variable X is a function $X: S \to \mathbb{R}$.

ex: Let S be the sample space consisting of all possible outcomes of 4 coin tosses. Let X be the number of heads that occur.

Q: What is $X(T, H, H, H)$? What is $X(T, T, H, T)$?

A: 1, 3

B: 2, 2

C: 3, 1

D: 4, 4

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def: The <u>expected</u> or <u>average</u> value of a random variable X is

$$\mathbb{E}[X] = \sum_{i \in S} \Pr(i)\, X(i).$$

Q: From previous example, what is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \sum_{\substack{i \in S: \\ X(i)=0}} \Pr(i) \cdot 0 + \sum_{\substack{i \in S \\ X(i)=1}} \Pr(i) + \sum_{\substack{i \in S \\ X(i)=2}} \Pr(i) \cdot 2$$

$$+ \sum_{\substack{i \in S \\ X(i)=3}} \Pr(i)\, 3 + \sum_{\substack{i \in S \\ X(i)=4}} \Pr(i)\, 4$$

$\Pr(i) = \frac{1}{2^4} = \frac{1}{16}$ in all cases.

$|\{i \in S : X(i) = 0\}| = 1$ $\qquad$ $|\{i \in S : X(i) = 2\}| = \binom{4}{2} = 6$

$|\{i \in S : X(i) = 1\}| = \binom{4}{1} = 4$ $\qquad$ $|\{i \in S : X(i) = 3\}| = \binom{4}{3} = 4$

$|\{i \in S : X(i) = 4\}| = 1$

$$\mathbb{E}[X] = \frac{1}{16}\left(1 \cdot 0 + 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 3 + 1 \cdot 4\right)$$

$$= \frac{1}{16}(32) = 2$$

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Indicator Random Variable:

def: An indicator random variable X is a random variable such that $X : S \to \{0,1\}$.

An indicator random variable is associated with an event $E \subseteq S$

$$E = \{i \in S : X(i) = 1\}$$

Normally write as X_E where

$$X_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbb{E}[X_E] = \sum_{i \in S} \Pr(i)\, X_E(i)$$

$$= \sum_{i \in E} \Pr(i) = \Pr(E)$$

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Linearity of Expectation

Let $Y_1, Y_2, \ldots, Y_n$ be random variables on a sample space S. Let $a_1, a_2, \ldots a_n \in \mathbb{R}$. Let Y be a random variable s.t.

$$Y = \sum_{k=1}^{n} a_k Y_k \qquad \left(\text{That is } \forall i \in S,\ Y(i) = \sum_{k=1}^{n} a_k Y_k(s)\right)$$

Then

$$\mathbb{E}[Y] = \sum_{k=1}^{n} a_k \mathbb{E}[Y_k]$$

Ex: Let X_k be the indicator random variable that takes value 1 if k^{th} coin flip is Heads.

$$X = \#\text{ of heads in 4 coin tosses}$$

Then $\quad X = \sum_{k=1}^{4} X_k$

e.g.

$$X(H,T,T,H) = \overset{1}{X_1(H,T,T,H)} + \overset{0}{X_2(H,T,T,H)} + \underset{0}{X_3(H,T,T,H)} + \underset{1}{X_4(H,T,T,H)} = 2$$

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Q: What is the expected # of heads if the coin is flipped 10 times?

$s \in \{T, H\}^{10}$

Let X_k be indicator random variable

$$X_k(s) = \begin{cases} 1 & \text{if } k^{th} \text{ flip of } s \text{ is heads} \\ 0 & \text{else} \end{cases}$$

Let X be random variable that is the total number of heads

Then:

$$X = \sum_{k=1}^{10} X_k$$

$$\mathbb{E}[X] = \sum_{k=1}^{10} \mathbb{E}[X_k] = \sum_{k=1}^{10} \Pr(k^{th} \text{ flip is heads})$$

$$= \sum_{k=1}^{10} \frac{1}{2} = 5$$

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Q: Consider $S = \{1, 2, 4\}^n$, where strings are chosen with uniform probability. What is the expected sum of digits in the string?

ex: 1 4 1 2 2 = 10

Define the following random variables:

$$X_{k,1} = \begin{cases} 1 & \text{if } k^{th} \text{ digit is } 1 \\ 0 & \text{else} \end{cases}$$

$$X_{k,2} = \begin{cases} 1 & \text{if } k^{th} \text{ digit is } 2 \\ 0 & \text{else} \end{cases}$$

$$X_{k,4} = \begin{cases} 1 & \text{if } k^{th} \text{ digit is } 4 \\ 0 & \text{else} \end{cases}$$

X = Sum of digits

Then

$$X = \sum_{k=1}^{n} X_{k,1} + 2X_{k,2} + 4X_{k,4}$$

Using Linearity of Expectation

$$\mathbb{E}[X] = \sum_{k=1}^{n} \mathbb{E}[X_{k,1}] + 2\,\mathbb{E}[X_{k,2}] + 4\,\mathbb{E}[X_{k,4}]$$

$$= n\left(\frac{1}{3} + \frac{2}{3} + \frac{4}{3}\right) = n \cdot \frac{7}{3}$$