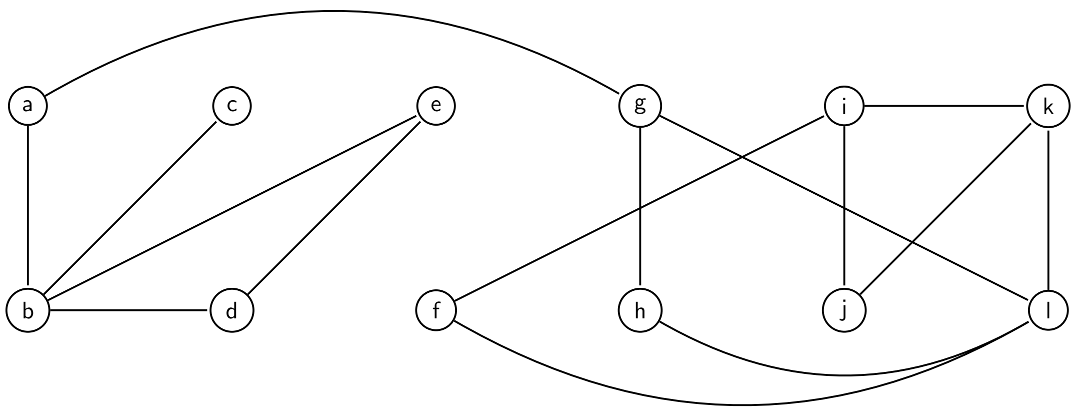
CS200 - Problem Set 9

Due: Monday, Nov. 20 to Canvas

1. Let G be the following graph, and let A_G be an adjacency list representation of the G with the adjacency list for each vertex in alphabetical order.



Consider the following slight variation to breadth-first-search:

Algorithm 1: `BFSish`(A, s)

Input : Adjacency list A for a graph (V, E) and vertex s

Output: An integer array L of length $|V|$.

`// Initialize array of explored vertices and array` L

$X[v] = 0\ \forall v \in V$;

$L[v] = \infty\ \forall v \in V$;

$X[s] = 1$;

$L[s] = 0$;

`// Initialize Queue` A

$A = \{\}$;

$A.add(s)$;

while *A is not empty* **do**

    $v = A.pop$;

    for *each edge* $\{v, w\}$ **do**

        if $X[w] == 0$ **then**

            $X[w] = 1$;

            $A.add(w)$;

            $L[w] = L[v] + 1$;

        end

    end

end

return L

(a) **[6 points]** In what order does `BFSish`(A_G, a) explore the nodes of the graph G? (Remember lists of the adjacency list representation are in alphabetical order, so the for loop in line 9 will look at vertices in alphabetical order.)

(b) **[6 points]** What are the values in the array L that is returned when `BFSish`(A_G, a) is implemented?

(c) **[3 points]** Considering your answer to part b, what does it seem like the algorithm is doing? What is the meaning of L? Think about the relationship between a, b, and $L[b]$.

2. Geometric Series

   (a) **[11 points]** Use induction to prove that for all $n \in \mathbb{N}$, and any $r \in \mathbb{R}$ such that $r \neq 0$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}. \qquad (1)$$

   (b) **[2 points]** What does the sum evaluate to when $r = 1$?

3. Recurrence Relations

   (a) Let $G(n)$ be the number of bit strings of length n that have two consecutive zeros. (A bit string of length n is an element of $\{0,1\}^n$.) Consider a recurrence relation for $G(n)$.

      i. **[3 points]** What are the initial conditions for the recurrence relation?

      ii. **[6 points]** What are the recursive conditions for the recurrence relation?

      iii. **[6 points]** Use the recurrence relation to calculate $T(5)$.

   (b) Consider the following variant to the Tower of Hanoi. We start with all disks on peg 1 and want ot move them to peg 3, but we cannot move a disk directly between peg 1 and peg 3. Instead, we can only move disks from peg 1 to peg 2, or from peg 2 to peg 3. So in the case of $n = 1$ disk, we have to first move the disk from 1 to 2, and then from 2 to 3. Let $T(n)$ be the number of moves required to shift a stack of n disks from peg 1 to peg 3, if as usual, we can not put a larger disk on top of a smaller disk. Consider creating a recurrence relation for $T(n)$.

      i. **[3 points]** What are the initial conditions for the recurrence relation?

      ii. **[6 points]** What are the recursive conditions for the recurrence relation? (Explain)

      iii. **[6 points]** Use the iterative method to solve for $T(n)$ and give a big-O bound on $T(n)$.

4. Consider rolling 5 dice. Let $X_{i,j}$ be an indicator random variable that takes value 1 if the ith di has outcome j (and takes value 0 otherwise).

   (a) **[3 points]** What is the sample space? What is its size?

   (b) **[3 points]** Let X be the random variable that is the sum of all of the values shown on the dice. If the outcome of the rolls is $s = (4, 2, 4, 5, 1)$, what is $X(s)$? What is $X_{3,4}(s)$? What is $X_{4,3}(s)$?

   (c) **[3 points]** Write X in terms of a *weighted* sum of the variables $X_{i,j}$.

(d) **[3 points]** What is $\mathbb{E}[X_{i,j}]$?

(e) **[3 points]** Use linearity of expectation to determine the average value of the sum of all values shown on the dice.

5. Suppose a group of n people each order a different flavor of ice cream at an ice cream shop. Suppose the server didn't keep track of who ordered which flavor, and just handed the ice cream out randomly.

   (a) **3 points** Let X a random variable that is the number of people who got handed the correct flavor. Let X_i be the indicator random variable that takes value 1 if person i gets the correct flavor (and 0 otherwise.) Write X in terms of a sum of the X_i.

   (b) **6 points** Use linearity of expectation to determine the average number of people who get the correct flavor.

6. How long did you spend on this homework?