Error Correction

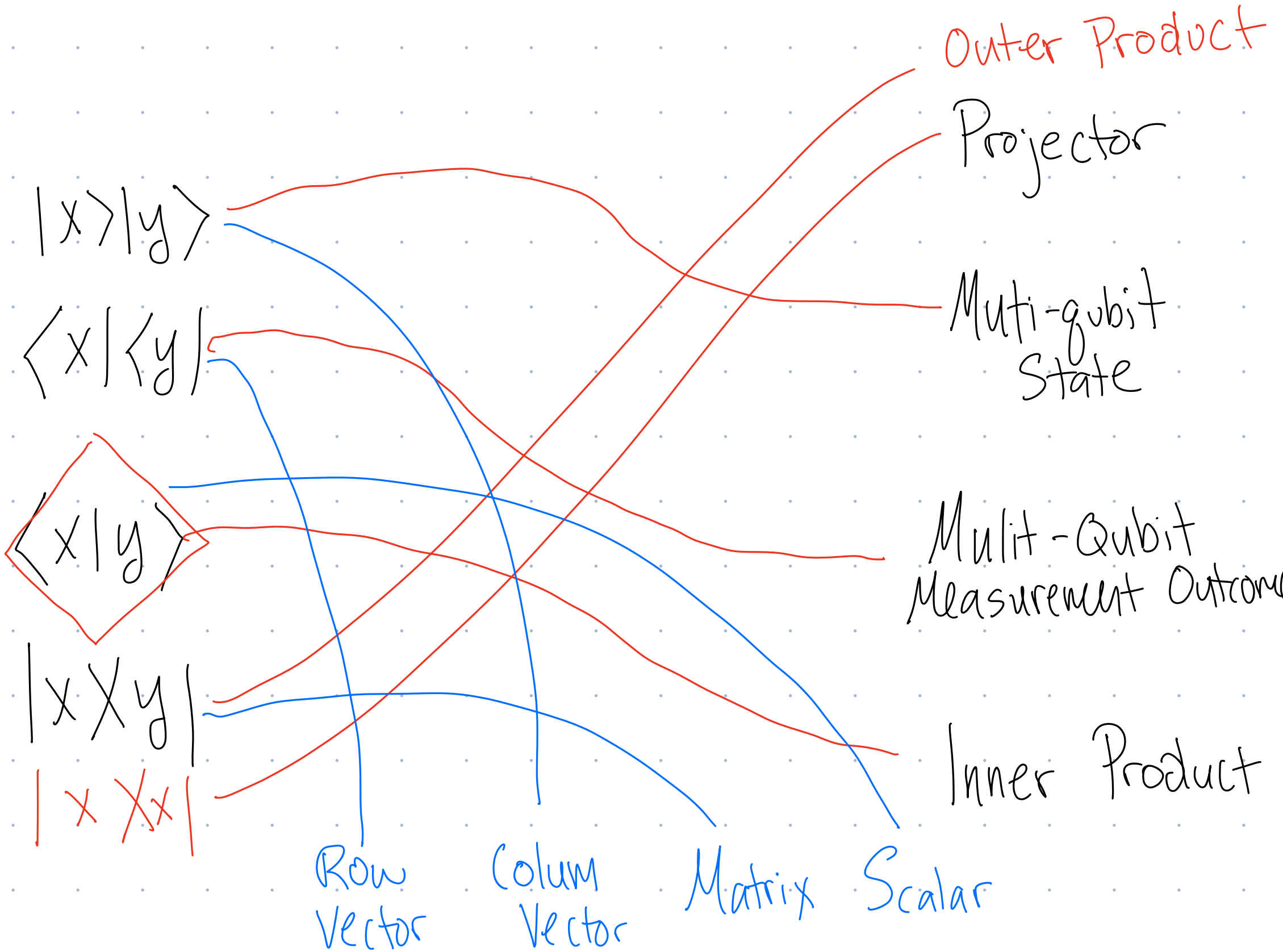
Learning Goals

- Describe errors in quantum/classical systems
- Describe repetition codes + why they work
- Analyze how a quantum repetition code corrects errors using projective partial measurements ← [QC4]

Announcements

- 2pm Friday: "Why Grad School?" with Andrew de Melo, Assistant Director of Grad Admissions @ UMASS Amherst

- Week from Thursday, In Class, Omar Armbruster '24(?), Director Operations at Quantum Rings


Outer Product
Projector
|x⟩|y⟩
⟨x|⟨y|
Muti-qubit State
⟨x|y⟩
Mulit-Qubit Measurement Outcome
|x⟩⟨y|
|x⟩⟨x|
Inner Product
Row Vector
Colum Vector
Matrix
Scalar

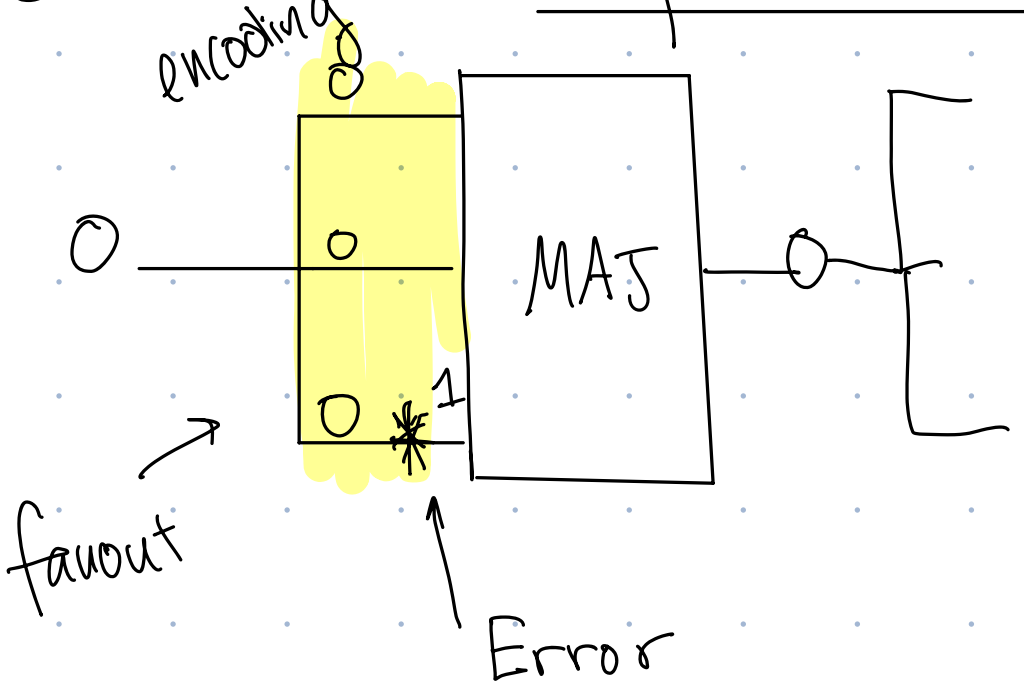
If Q. Computers are so great, why haven't we built one?

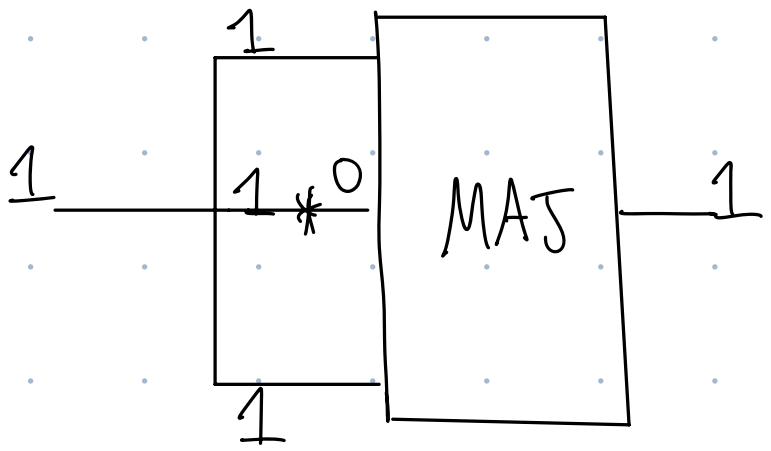
ERRORS - unwanted gates / measurement

- Classical Computers
  - Cosmic rays: largest source of error
    ↳ 1 bit flip / 4 GB per day
    ↓
    so small that usually ignored

- Pace maker - vote counting machine - Plane drop
- Super mario glitch

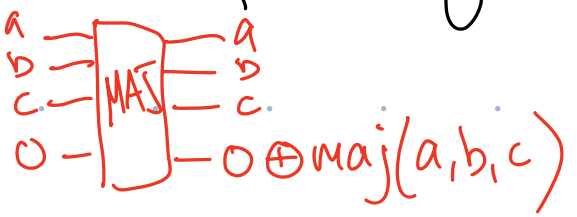
Solution: Repetition Code





- Sources of Quantum Errors
  - control lasers not perfect
    (shape, frequency, focus, intensity)
  - imperfect vacuum
  - Stray magnetic, electric fields
  - Heat (anomalous heating)

Why won't repetition code work for quantum?



A) MAJ not reversible

B) Doesn't correct phase errors (Z)

C) No fanout because can't copy quant

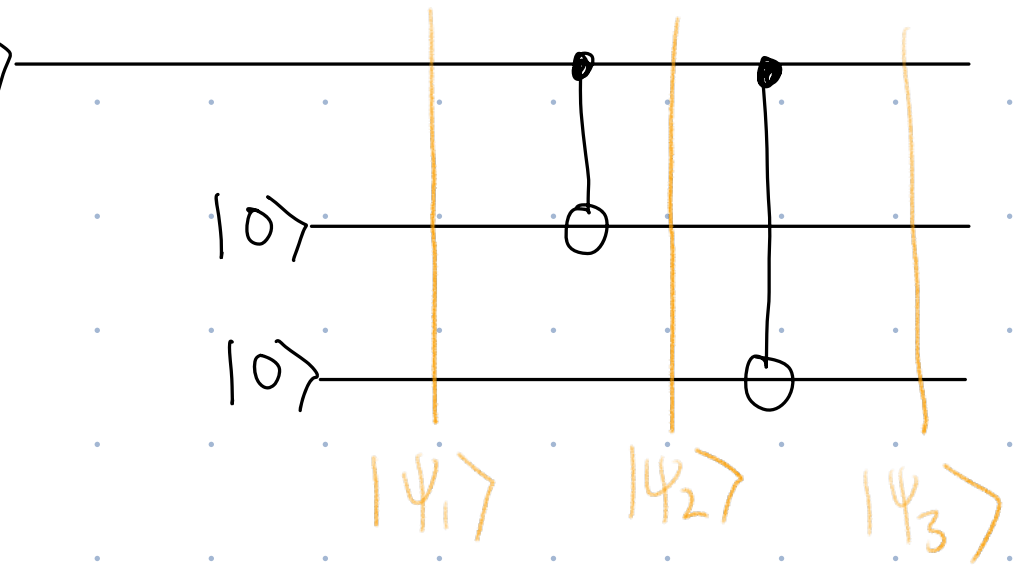
D) There are uncountably infinitely many errors to deal with

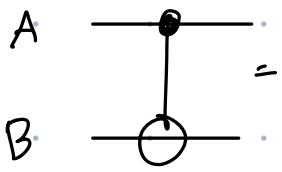
Some scientists think error correction at a large scale is impossible

Shor's Repetition Code

Encoding:

$|\psi\rangle = a|0\rangle + b|1\rangle$





= CNOT

If A = |1⟩,

→ X B

$|\psi_1\rangle = (a|0\rangle + b|1\rangle)|0\rangle|0\rangle$

$|\psi_2\rangle$

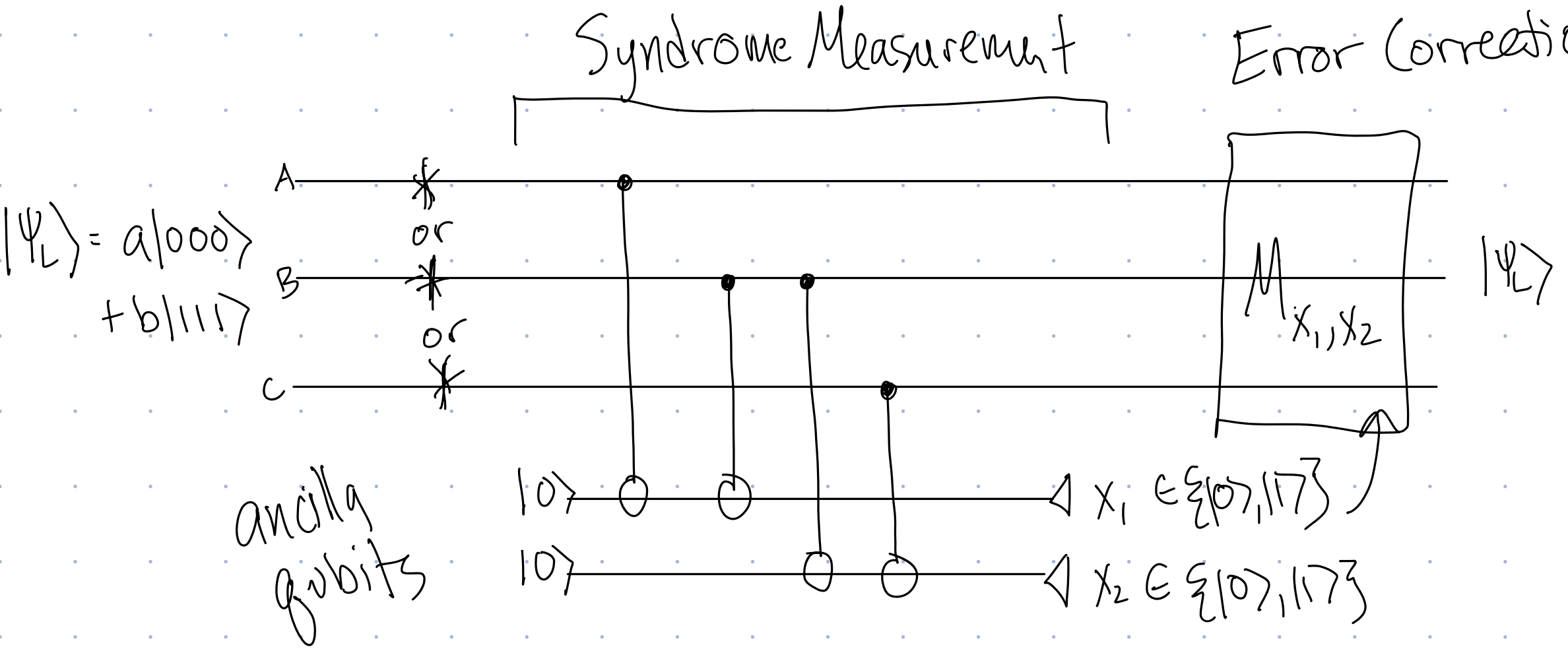
$|\psi_3\rangle = a|000\rangle + b|111\rangle = |\psi_L\rangle$

"Logical Qubit"

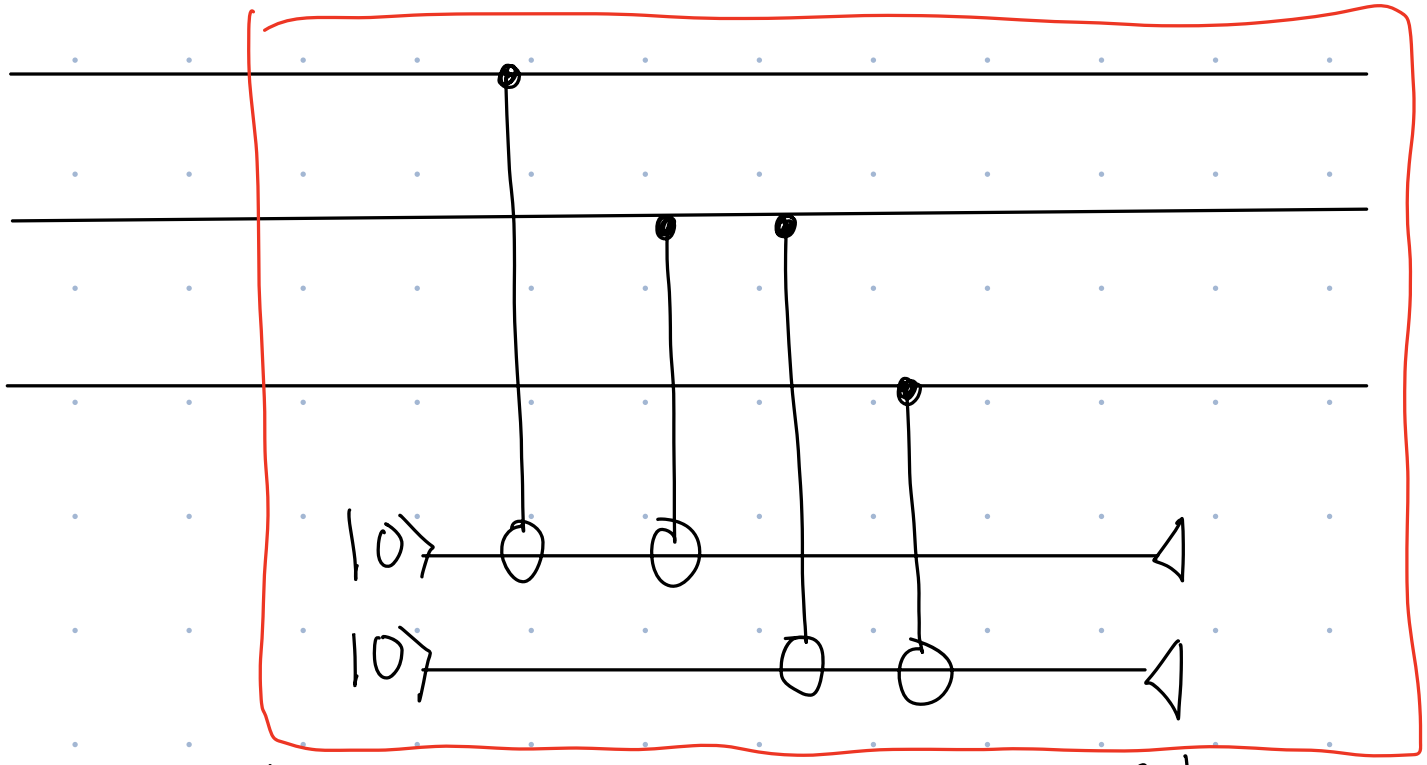
vs

"Physical Qubit"

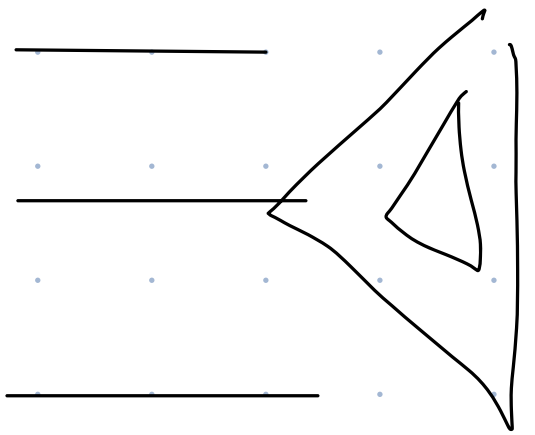
Error Correction Circuit



Syndrome Meas.



≈



Not another partial measurement!

New approach: Projective Measurement

Projective Measurement

Described by a Set of projectors $M = \{P_0, P_1, P_2, \dots\}$

$$P_0 = |\phi_1\rangle\langle\phi_1| + |\phi_4\rangle\langle\phi_4|$$

$$P_1 = |\phi_3\rangle\langle\phi_3|$$

$$P_2 = |\phi_2\rangle\langle\phi_2| + |\phi_5\rangle\langle\phi_5| + |\phi_6\rangle\langle\phi_6|$$

$$\vdots$$

$|\phi_i\rangle$s should be orthornormal

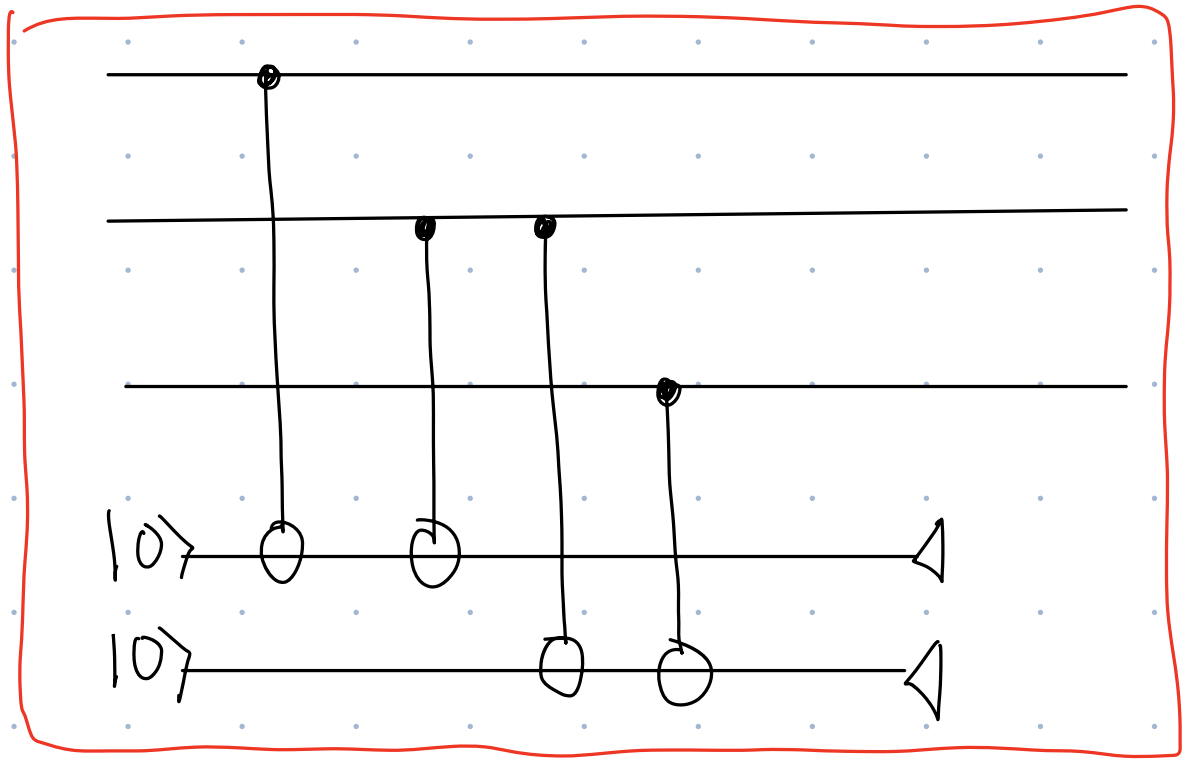
$|\phi_1\rangle\langle\phi_1|$

$|\phi\rangle\langle\phi|$

techical: $\sum_i P_i = I$

Probability of outcome P_i ← when measuring $|\psi\rangle$: $\Pr(P_i) = \langle\psi|P_i|\psi\rangle$

After outcome i, State collapses to $\dfrac{P_i|\psi\rangle}{\sqrt{\Pr(P_i)}}$



$|+X+|$

P_0

$$M = \{\ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,\ |010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011|\ \}$$

P_1 P_2 P_3

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle$ with M, which outcomes are possible? $|\psi\rangle \to \frac{P_i|\psi\rangle}{\sqrt{Pr(i)}}$ $Pr(i) = \langle\psi|P_i|\psi\rangle$

A) P_0
B) P_1
C) P_2
D) P_3

$$\left(\tfrac{1}{\sqrt{6}}\langle 000| + \sqrt{\tfrac{2}{6}}\langle 001| + \sqrt{\tfrac{3}{6}}\langle 110|\right)\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)\left(\tfrac{1}{\sqrt{6}}|000\rangle + \sqrt{\tfrac{2}{6}}|001\rangle + \sqrt{\tfrac{3}{6}}|110\rangle\right)$$

$$\tfrac{1}{\sqrt{6}}\tfrac{1}{\sqrt{6}}\langle 000|000\rangle\langle 000|000\rangle + \sqrt{\tfrac{2}{6}}\cdot\sqrt{\tfrac{3}{6}}\langle 001|111\rangle\langle 111|110\rangle$$

$1 \quad 1 \quad 0 \quad 0$

$\frac{1}{6}$ 0

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle$ with M,

and get outcome $P_1 = |001\rangle\langle 001| + |110\rangle\langle 110|$ what is

the resultant state? $|\psi\rangle \to \frac{P_i|\psi\rangle}{\sqrt{Pr(i)}}$ $Pr(i) = \langle\psi|P_i|\psi\rangle$

$$\frac{1}{\sqrt{Pr}}\left(|001\rangle\langle 001| + |110\rangle\langle 110|\right)\left(\frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)$$

$$\frac{1}{\sqrt{2}}|001\rangle\langle 001|000\rangle + \cdots$$

$$= \frac{1}{\sqrt{Pr}}\left(\sqrt{\frac{2}{6}}|001\rangle\langle 001|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\langle 110|110\rangle\right)$$

$$= \frac{1}{\sqrt{Pr}}\left(\sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)$$

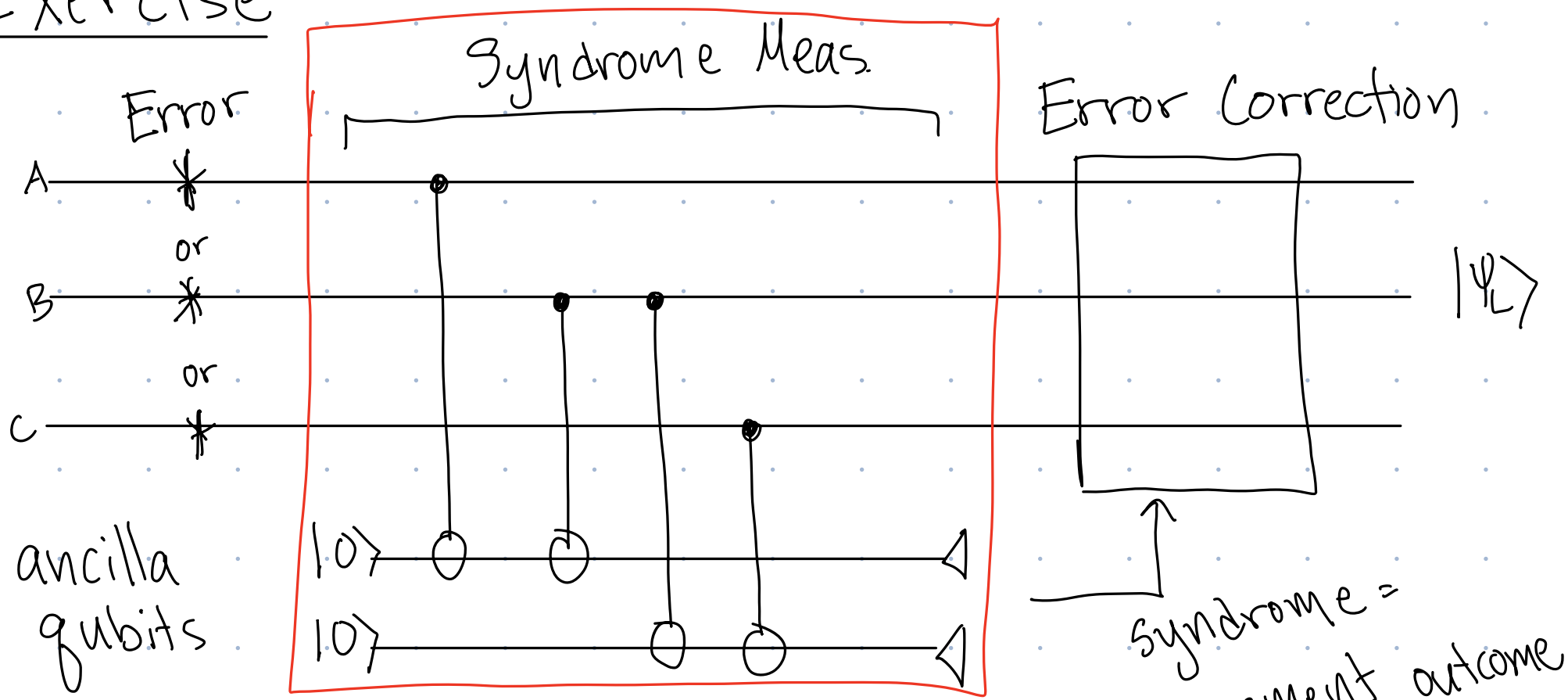
$$= \sqrt{\frac{6}{5}}\left(\sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right) = \sqrt{\frac{2}{5}}|001\rangle + \sqrt{\frac{3}{5}}|110\rangle$$

$Pr = \frac{5}{6}$

Group Exercise

$a|0\rangle + b|1\rangle$

$|\psi_L\rangle = a|000\rangle + b|111\rangle$



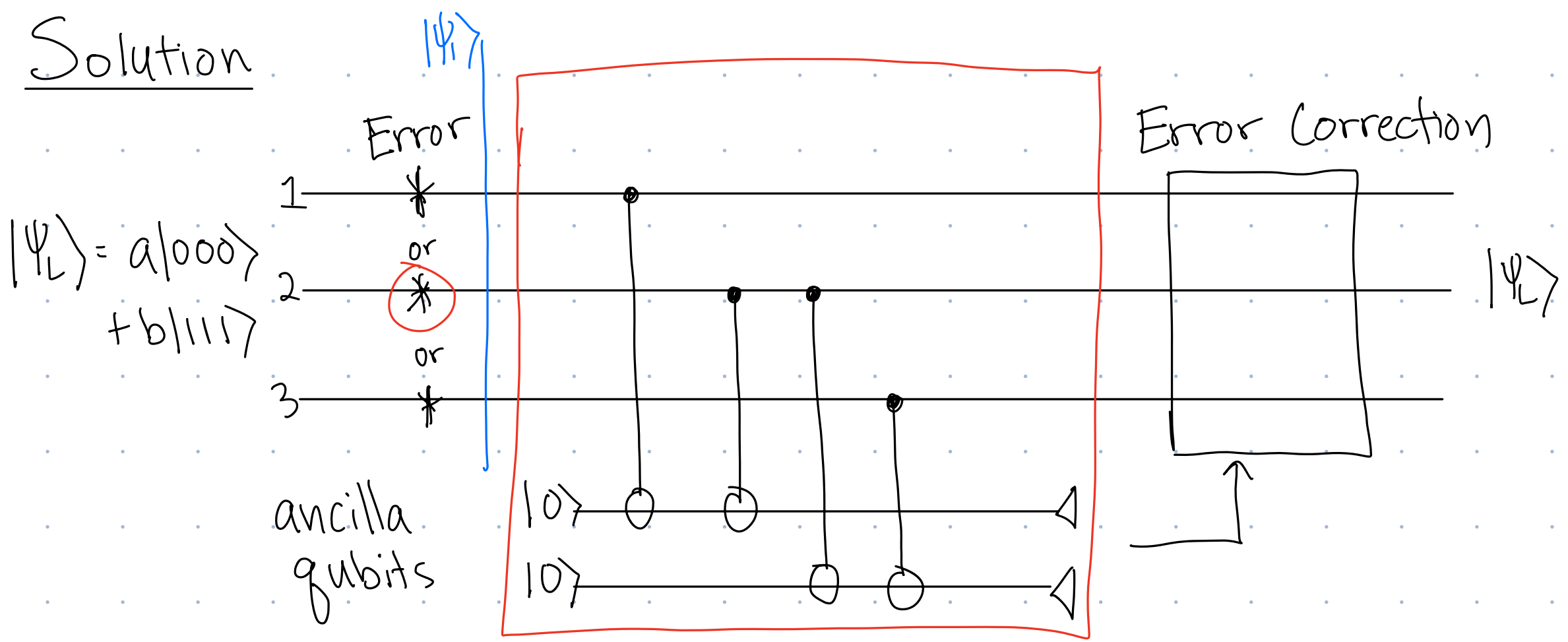
$M = \{|000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,$
$|010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011|\}$

*: $R_x(\theta) = \cos\theta I + i\sin\theta X$

ex: $R_x(\theta)|1\rangle = \cos\theta I|1\rangle + i\sin\theta X|1\rangle = \cos\theta|1\rangle + i\sin\theta|0\rangle$

1. What are possible outcomes + what does state collapse to?
2. What error correction procedure should be used in each case?

Solution



$$M = \{|000X000| + |111X111|,\ |001X001| + |110X110|,$$
$$|010X010| + |101X101|,\ |100X100| + |011X011|\}$$

$$|\psi_1\rangle = (\cos\theta I + i\sin\theta X_2)(a|000\rangle + b|111\rangle)$$
$$= a\cos\theta|000\rangle + b\cos\theta|111\rangle + ai\sin\theta|010\rangle + bi\sin\theta|101\rangle$$

- Outcome P_0 : $\Pr(P_0) = \langle\psi_1|P_0|\psi_1\rangle$

$P_0|\psi_1\rangle$

$$\left(a^*\cos\theta\langle 000| + b^*\cos\theta\langle 111|\right)\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)\left(a\cos\theta|000\rangle + b\cos\theta|111\rangle\right)$$

$$\left(a\cos\theta|000\rangle + b\cos\theta|111\rangle\right)$$

$$= a^*a\cos^2\theta\langle 000|000\rangle + b^*b\cos\theta\langle 111|111\rangle$$

$$= \left(|a|^2 + |b|^2\right)\cos^2\theta$$

$$= \cos^2\theta$$

$$|\psi\rangle_{collapsed} = \frac{1}{\cos\theta}\left(a\cos\theta|000\rangle + b\cos\theta|111\rangle\right)$$

$$= a|000\rangle + b|111\rangle$$

- Outcome P_2

  Prob: $\sin^2\theta$

  $|\psi\rangle_{collapse} \rightarrow a|010\rangle + b|101\rangle$ (i global phase)

To fix: Apply X_2 (X on middle qubit)

Error on 1 qubit: Outcomes P_0, $P_3 \rightarrow X_1$ to fix

Error on 2 qubit Outcomes P_0, $P_2 \rightarrow X_2$ to fix

Error on 3 qubit Outcomes P_0, $P_1 \rightarrow X_3$ to fix

↓

Collapse back to $|\psi_L\rangle$,
the original state,
No active E.C. needed

Big Idea with Error Correction

Correction Rule: Use lowest weight correction

ex: Get outcome $\{|100X100| + |011X011|\}$ in 3 qubit Shor code, so collapse $\Rightarrow$ $|\psi\rangle \to \alpha|100\rangle + \beta|011\rangle$ (unknown amplitudes)

- If apply $X_1 \Rightarrow$

- If apply $X_2X_3 \Rightarrow$

Need to Correct More Errors? Bigger Code!

$$|0_L\rangle = (|000\rangle_{123} + |111\rangle_{123})(|000\rangle_{456} + |111\rangle_{456})(|000\rangle_{789} + |111\rangle_{789})$$
$$|1_L\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

} 9-qubit Shor Code

↑

Corrects any single qubit error

$$|0_L\rangle = (|000\rangle_{123} + |111\rangle_{123})(|000\rangle_{456} + |111\rangle_{456})(|000\rangle_{789} + |111\rangle_{789})$$
$$|1_L\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

$$|0_L\rangle = (|000\rangle_{123} + |111\rangle_{123})(|000\rangle_{456} + |111\rangle_{456})(|000\rangle_{789} + |111\rangle_{789})$$
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Need to Correct More Errors? Bigger Code!

$$|0_L\rangle = (|000\rangle_{123} + |111\rangle_{123})(|000\rangle_{456} + |111\rangle_{456})(|000\rangle_{789} + |111\rangle_{789})$$

$$|1_L\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Lowest weight way to return to code space?

Which errors can be corrected by Shor's code?

$$|0_L\rangle = (|000\rangle_{123} + |111\rangle_{123})(|000\rangle_{456} + |111\rangle_{456})(|000\rangle_{789} + |111\rangle_{789})$$

$$|1_L\rangle = (|000\rangle_{123} - |111\rangle_{123})(|000\rangle_{456} - |111\rangle_{456})(|000\rangle_{789} - |111\rangle_{789})$$

A) $Z_1 Z_2$

B) $Z_4 Z_8$

C) $Z_1 Z_2 Z_3$

D) $X_1 Z_2$

E) $X_1 X_2$

F) $X_1 X_4$

G) $Y_1 Z_4$

Fault Tolerant Quantum Computing

Shor Code: All single qubit errors, 9 qubits, 1 logical qubit

Concatenation:

$$|0_L\rangle = (|000\rangle_{123} + |111\rangle_{123})(|000\rangle_{456} + |111\rangle_{456})(|000\rangle_{789} + |111\rangle_{789})$$

$$|1_L\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

But... more qubits means more chance of error. Can only keep ahead of accumulating errors if error rate is small enough

Each code has an error threshold they can need to concatenate without accumulating errors.



Papers with more info

https://images.app.goo.gl/vJubz7fLrgNLWdNQ9

https://nam02.safelinks.protection.outlook.com/?url=https%3A%2F%2Farxiv.org%2Fhtml%2F2402.11105v2&data=05%7C02%7Cskimmel%40middlebury.edu%7C9463e5e57f11493ed1b508dd08c71f7b%7Ca1bb0a191576421dbe93b3a7d4b6dcaa%7C1%7C0%7C638676373508901760%7CUnknown%7CTWFpbGZsb3d8eyJFbXB0eU1hcGkiOnRydWUsIlYiOiIwLjAuMDAwMCIsIlAiOiJXaW4zMiIsIkFOIjoiTWFpbCIsIldUIjoyfQ%3D%3D%7C0%7C%7C%7C&sdata=Kld4x28zXhYebWsLlRdkYD%2FstF9TnC2n5IyyORc2CiE%3D&reserved=0