Error Correction

Learning Goals

- Describe errors in quantum/classical systems
- Describe repetition codes + why they work
- Analyze how a repetition code corrects errors

Announcements

- Discussion Next Week - Hype
- Project Due Dates
- MiddDev Soft. Eng. Alum Talk: Sun @ 1 in 75SHS 203.
- JTerm Workshop on Leetcode + coding interview prep

Exit Tickets

- Partial to Projective → in situ
- Continuous versus discrete error

Do we have to know what error occured to correct it?

A) Yes

B) No

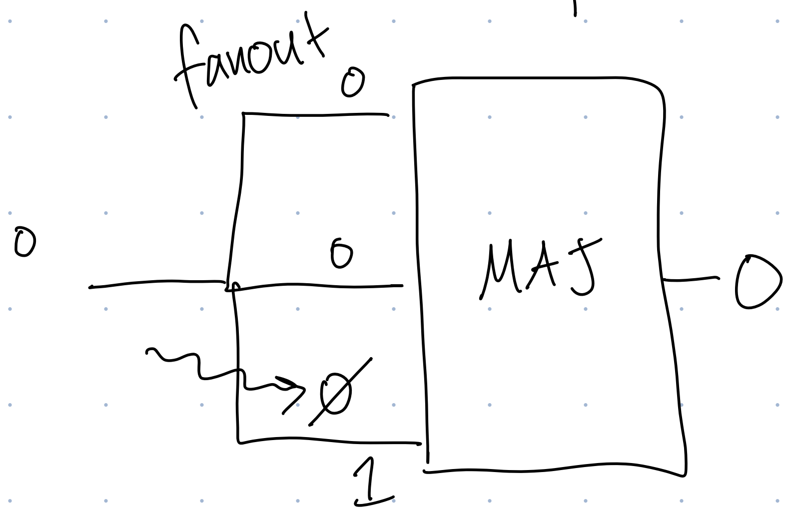
C) Kind Of

If Q. Computers are so great, why haven't we built one?

Errors - unwanted gates/measurements

- Classical Computers
  - cosmic rays "soft errors"
    ↳ 1 bit flip / 4 GB per day

Solution: <u>Repetition Code</u>



- Sources of Quantum Errors
  - control lasers not perfect (shape, frequency, focus, intensity)
  - imperfect vacuum
  - Stray magnetic, electric fields
  - Heat (anomalous heating)

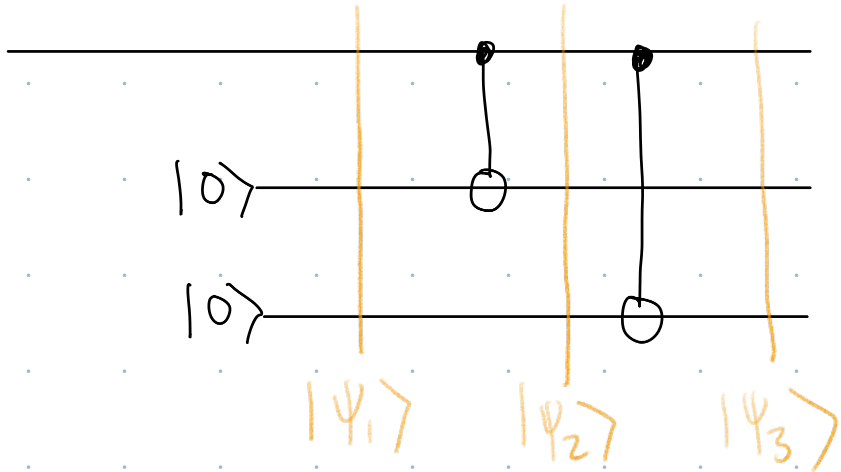
Why won't repetition code work for quantum?

A) MAJ not reversible
B) Doesn't correct phase errors (Z)
C) No fanout because can't copy quantum states
D) There are uncountably infinitely many errors to deal with

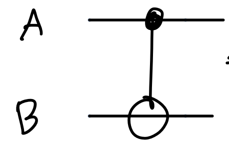
Shor's Repetition Code

Encoding:

$|\psi\rangle = a|0\rangle + b|1\rangle$



↑ original state to protect from errors. Don't know a,b.



= CNOT

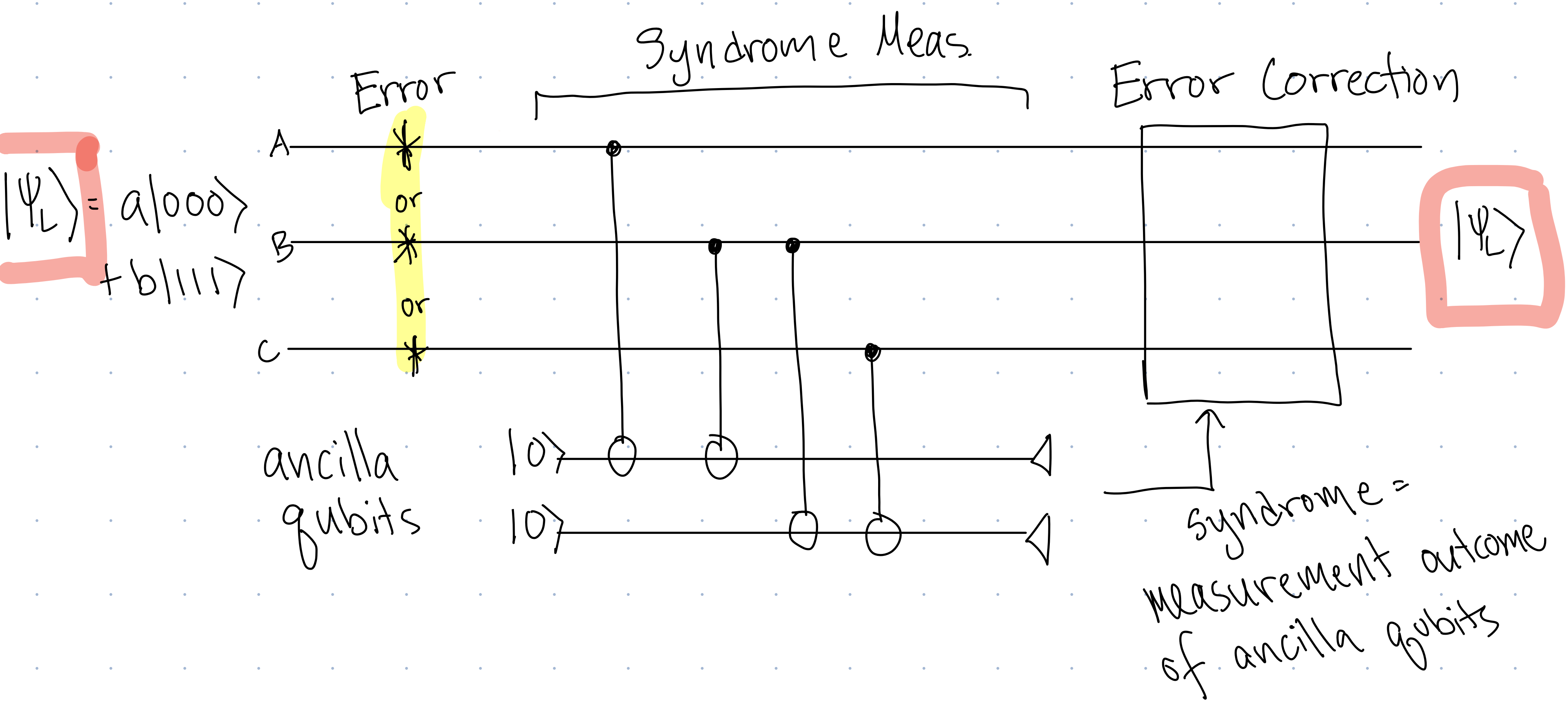
If $A = |1\rangle$, → $X\,B$

$$|\psi_1\rangle = \big(a|0\rangle + b|1\rangle\big)|0\rangle|0\rangle = a|000\rangle + b|100\rangle$$

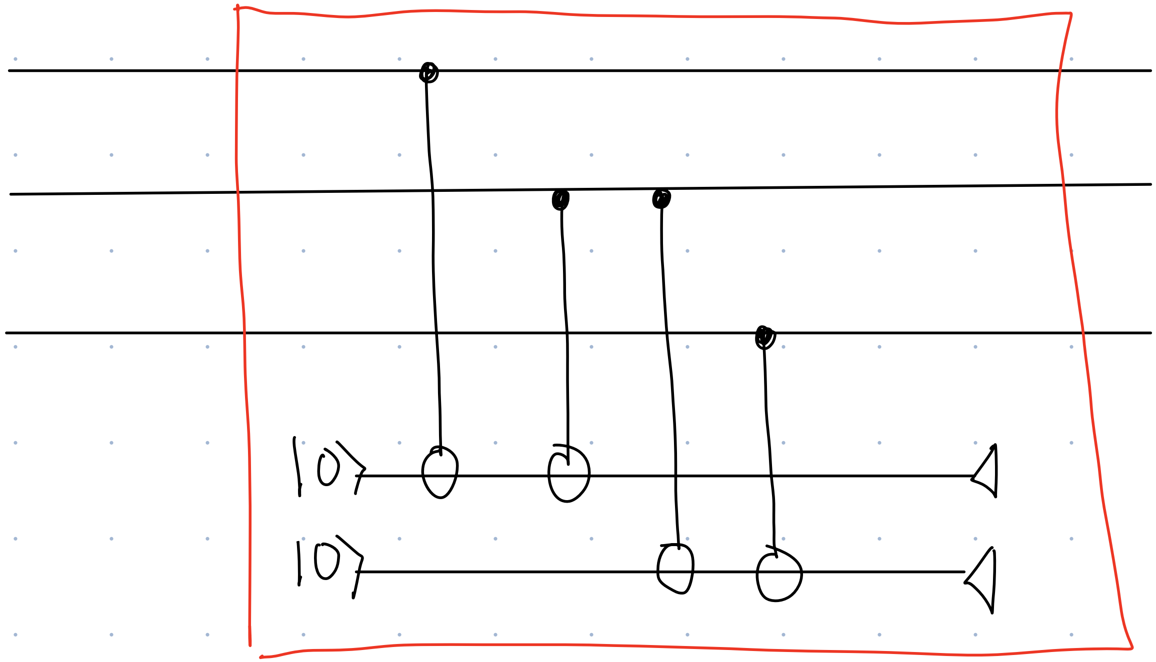
$$|\psi_2\rangle = a\,CNOT_{12}|000\rangle + b\,CNOT_{12}|100\rangle$$
$$= a|000\rangle + b|110\rangle$$

$$|\psi_3\rangle = a\,CNOT_{13}|000\rangle + b\,CNOT_{13}|110\rangle$$
$$= a|000\rangle + b|111\rangle$$

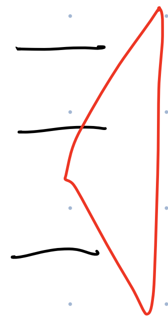
Error Correction Circuit



Syndrome Meas. Projective Meas.



$\approx$



Projective Measurement

Described by a set of projectors $M = \{P_0, P_1, P_2, \dots\}$

$\uparrow$ outcomes

ex: $P_0 = |\phi_1\rangle\langle\phi_1| + |\phi_4\rangle\langle\phi_4|$

$P_1 = |\phi_3\rangle\langle\phi_3|$

$P_2 = |\phi_2\rangle\langle\phi_2| + |\phi_5\rangle\langle\phi_5| + |\phi_6\rangle\langle\phi_6|$

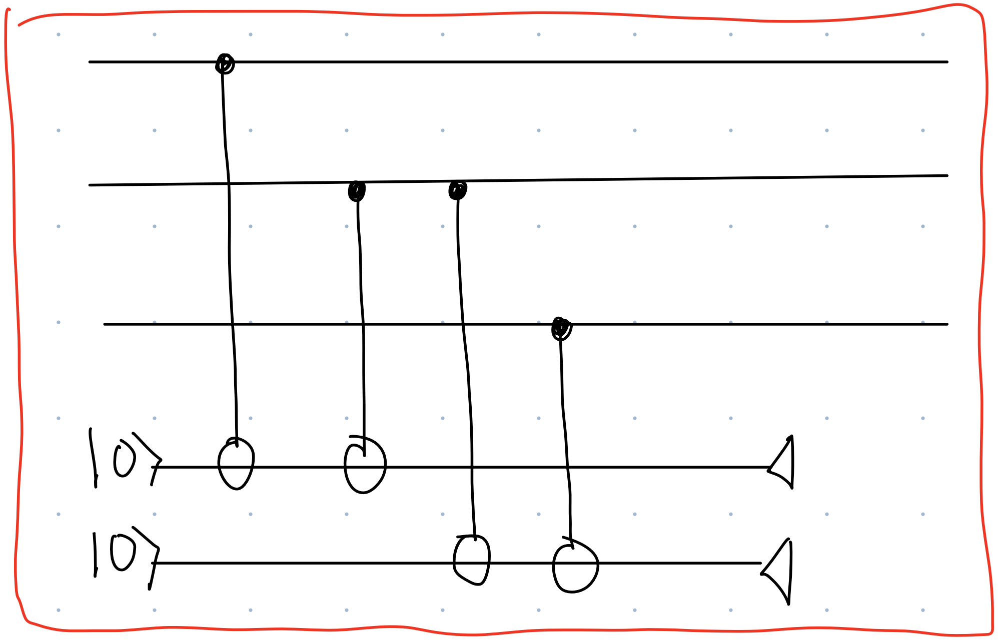
$\boxed{|\phi\rangle\langle\phi| \text{ write as } |\phi\rangle\langle\phi|}$

- $|\phi_i\rangle$ should be N-dimensional, where N is dim. of system being measured
- Technical $\sum P_i = I$

If measure $|\psi\rangle$

$\rightarrow$ Probability of outcome P_i: $\langle\psi|P_i|\psi\rangle$

$\rightarrow$ If outcome i: collapse to $\dfrac{P_i|\psi\rangle}{\sqrt{Pr(i) = \langle\psi|P_i|\psi\rangle}}$



$$\approx \quad M = \{ \overbrace{|000\rangle\langle 000| + |111\rangle\langle 111|}^{P_0}, \; \overbrace{|001\rangle\langle 001| + |110\rangle\langle 110|}^{P_1}, \; \overbrace{|010\rangle\langle 010| + |101\rangle\langle 101|}^{P_2}, \; \underbrace{|100\rangle\langle 100| + |011\rangle\langle 11|}_{P_3} \}$$

$$\langle \psi | P | \psi \rangle$$

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}\,|001\rangle + \sqrt{\frac{3}{6}}\,|110\rangle$ with M, which outcomes are possible?

$\boxed{\text{A) } P_0 \quad \text{B) } P_1}$

C) P_2

D) P_3

$$\big(\langle\psi|\big)\overbrace{\big(|000\rangle\langle 000| + |111\rangle\langle 111|\big)}^{P_0}\overbrace{\left(\frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)}^{|\psi\rangle}$$

$$= \langle\psi|\left(\frac{1}{\sqrt{6}}|000\rangle\underbrace{\langle 000|000\rangle}_{1} + \cancel{\frac{1}{\sqrt{6}}|111\rangle\underbrace{\langle 111|000\rangle}_{0}} + \cdots\right)$$

$$= \left(\frac{1}{\sqrt{6}}\langle 000| + \cancel{\sqrt{\frac{2}{6}}\langle 001|} + \cancel{\sqrt{\frac{3}{6}}\langle 110|}\right)\left(\frac{1}{\sqrt{6}}|000\rangle\right) = \frac{1}{6}$$

Prob of P

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}\,|001\rangle + \sqrt{\frac{3}{6}}\,|110\rangle$ with M and get outcome $P_1 = |001\rangle\langle 001| + |110\rangle\langle 110|$ what is the resultant state?

$$|\psi\rangle \to \frac{P_i|\psi\rangle}{\sqrt{Pr(i)}} \qquad Pr(i) = \langle\psi|P_i|\psi\rangle$$

$$P_i|\psi\rangle = \left(|001\rangle\langle 001| + |110\rangle\langle 110|\right)\left(\frac{1}{\sqrt{6}}|00\rangle + \sqrt{\frac{2}{6}}\,|001\rangle + \sqrt{\frac{3}{6}}\,|110\rangle\right)$$

$$Pr(P_1) = \frac{5}{6}$$

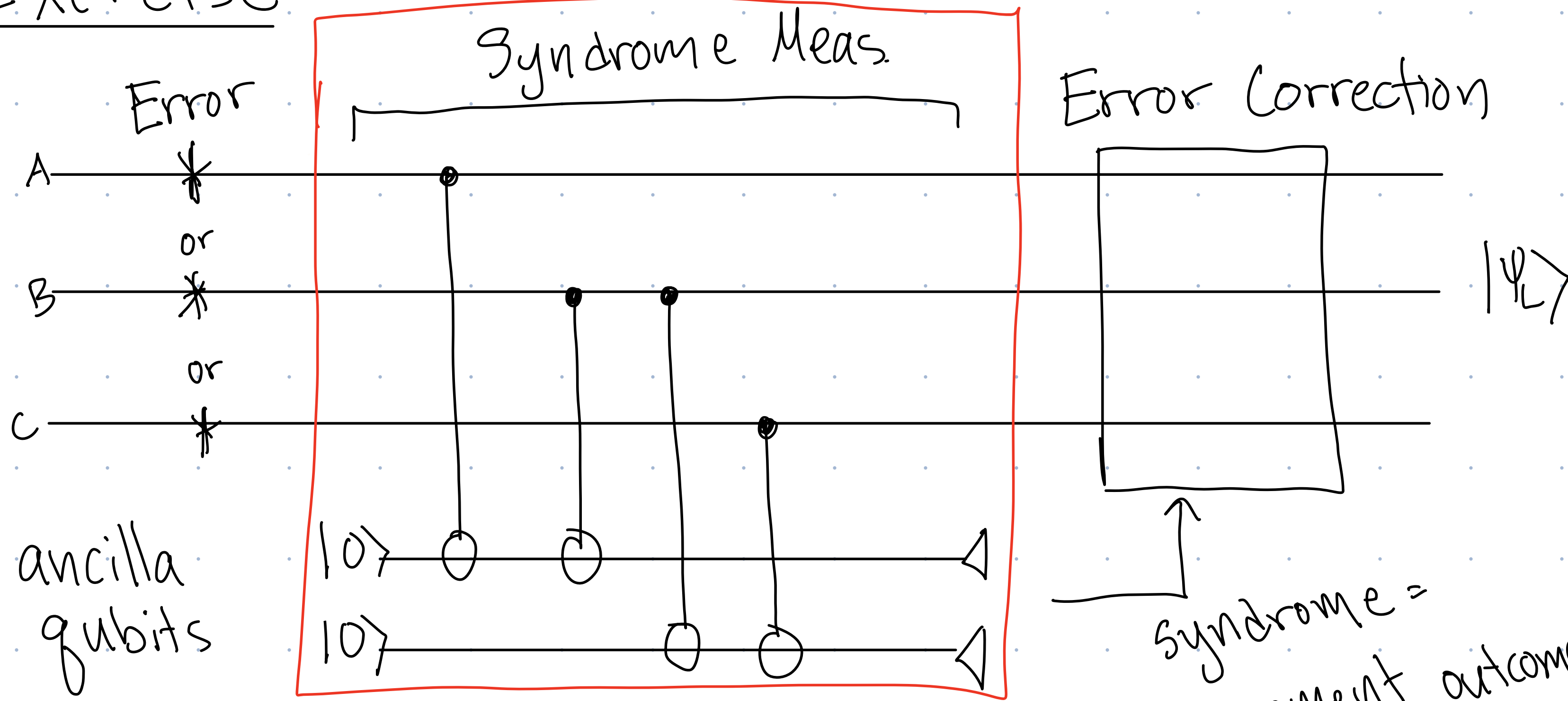
$$\sqrt{\frac{2}{6}}\,|001\rangle + \sqrt{\frac{3}{6}}\,|110\rangle\underbrace{\langle 110|110\rangle}_{1}$$

$$\to \frac{\sqrt{\frac{2}{6}}\,|001\rangle + \sqrt{\frac{3}{6}}\,|110\rangle}{\sqrt{\frac{5}{6}}} = \sqrt{\frac{2}{5}}\,|001\rangle + \sqrt{\frac{3}{5}}\,|110\rangle$$

Group Exercise

$|\psi_L\rangle = a|000\rangle + b|111\rangle$

$a|0\rangle + b|1\rangle$



syndrome = measurement outcome of ancilla qubits

$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,$
$|010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$

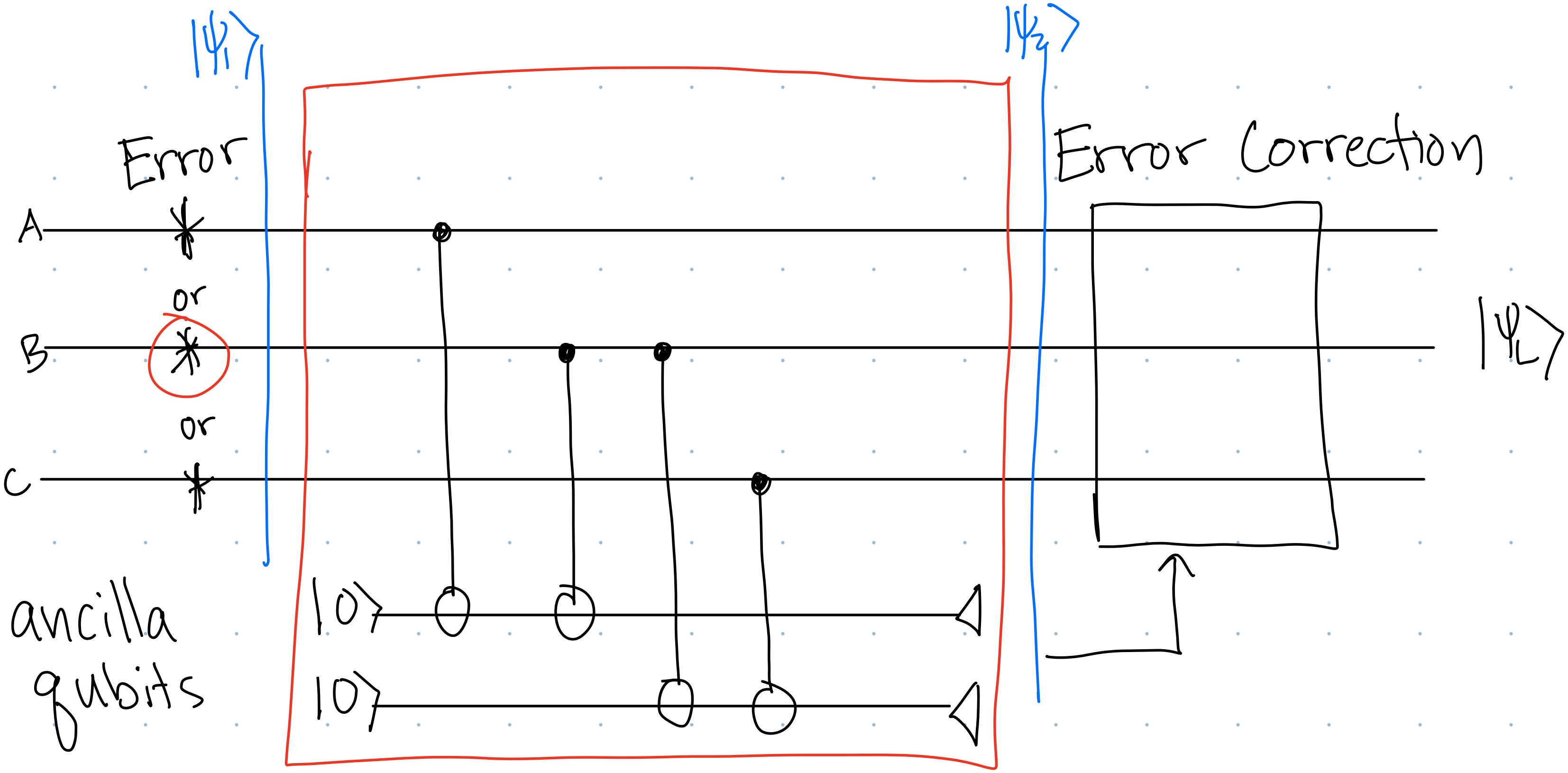
$*$: $R_x(\theta) = \cos\theta\, I + i\sin\theta\, X$ (continuous) $\quad I, X$ (discrete)

ex: $R_x(\theta)|1\rangle = \cos\theta\, I|1\rangle + i\sin\theta\, X|1\rangle = \cos\theta|1\rangle + i\sin\theta|0\rangle$

1. What are possible outcomes + what does state collapse to?
2. What error correction procedure should be used in each case?

Solution

$|\psi_L\rangle = a|000\rangle + b|111\rangle$



$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,$
$|010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$

$a|000\rangle \rightarrow a\cos\theta|000\rangle + a i \sin\theta|010\rangle$

$|\psi_1\rangle = a\cos\theta|000\rangle + a i\sin\theta|010\rangle + b\cos\theta|111\rangle + b i\sin\theta|101\rangle$

$=$

$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,$
$|010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$

$a|000\rangle \to a\cos\theta|000\rangle + ai\sin\theta|010\rangle$

$|\psi_1\rangle = a\cos\theta|000\rangle + ai\sin\theta|010\rangle + b\cos\theta|111\rangle + bi\sin\theta|101\rangle$

P_0: $\boxed{P_0|\psi_1\rangle = a\cos\theta|000\rangle + b\cos\theta|111\rangle}$

$\langle\psi_1|P_0|\psi_1\rangle = (a^*\cos\theta\langle 000| - a^* i\sin\theta\langle 010| + b^*\cos\theta\langle 111| - b^* i\sin\theta\langle 101|)$
$(a\cos\theta|000\rangle + b\cos\theta|111\rangle)$

$= a^*a\cos^2\theta + b^*b\cos^2\theta = \cos^2\theta(\underbrace{|a|^2 + |b|^2}_{1})$

$= \boxed{\cos^2\theta = \text{Prob of } P_0}$

Collapsed State:

$$\boxed{\frac{a\cos\theta|000\rangle + b\cos\theta|111\rangle}{\sqrt{\cos^2\theta}} = a|000\rangle + b|111\rangle}$$

P_2: $P_2|\psi_1\rangle = ai\sin\theta|010\rangle + bi\sin\theta|101\rangle$ Prob: $\sin^2\theta$

Collapsed State $\dfrac{P_2|\psi_1\rangle}{\sqrt{\sin^2\theta}} = ai|010\rangle + bi|101\rangle$

Apply $-iX_3 \Rightarrow a|000\rangle + b|111\rangle$

Error on A qubit: P_0 or P_3 $\Rightarrow$ X_A

Error on B qubit: P_0 or P_2 $\Rightarrow$ X_B

Error on C qubit: P_0 or P_1 $\Rightarrow$ $\underbrace{X_C \qquad}_{\text{Correct}}$ $\Big\}\ a|000\rangle + b|111\rangle$

$\Downarrow$

Collapse $a|000\rangle + b|111\rangle$

Quantum

Big Idea with Error Correction

- Partial collapse turns continuous error $R_x(\theta)$ into discrete error (X, I)
- Partial meas. provides info on how to correct while preserving superposition

Rules for Correcting Proj. Meas

$$P_1 = \{ |\psi_a\rangle\langle\psi_a| + |\psi_b\rangle\langle\psi_b| \}$$

$$\approx \{ |\psi_a\rangle, |\psi_b\rangle \}$$

If get outcome P_1, state collapses to

$$|\psi\rangle \rightarrow \alpha|\psi_a\rangle + \beta|\psi_b\rangle$$

We want to be in

$$\gamma|0_L\rangle + \eta|1_L\rangle$$

Correct by finding lowest weight correction

$P_1|\psi\rangle$

$$P_1 = \{ |011\rangle\langle 011| + |100\rangle\langle 100| \}$$

$$\uparrow$$

ex: Get Outcome $\{|011\rangle, |100\rangle\}$

Goal: $a|000\rangle$
$b|111\rangle$

$$|\psi\rangle \to \overset{b\;a}{\alpha}|011\rangle + \overset{a\;b}{\beta}|100\rangle$$

Lowest wt. correction

If apply X_1 $\to \alpha|111\rangle + \beta|000\rangle$

$\alpha|1_L\rangle + \beta|0_L\rangle$ ✓ Returned to codespace

If apply $X_2 X_3 \to \alpha|000\rangle + \beta|111\rangle$

$\alpha|0_L\rangle + \beta|1_L\rangle$ ✓ Returned to codespace

Need to Correct More Errors? Bigger Code!

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

9-qubit Shor Code

Corrects any single qubit error

$$P_{X_1} = \{X_1|0_L\rangle, X_L|1_L\rangle\}$$

$$X_1|0_L\rangle = \left(|\overset{1}{\cancel{0}}00\rangle_{123} + |\overset{0}{\cancel{1}}11\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$X_1|1_L\rangle = \left(|\overset{1}{\cancel{0}}00\rangle_{123} - |\overset{0}{\cancel{1}}11\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

$$P_{Z_6} = \{Z_6|0_L\rangle, Z_6|1_L\rangle\}$$

$$Z_6|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} \overset{\overset{+}{\cancel{-}}}{\cancel{+}} |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$Z_6|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} \overset{+}{\cancel{-}} |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

Need to Correct More Errors? Bigger Code!

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9-qubit Shor Code

Corrects any single qubit error

Z_6 error:

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

Z_1 and Z_4

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

Which errors can be corrected by Shor's code?

$a|0_L\rangle + b|0_L\rangle$

$$a|0_L\rangle = \left(|\overset{1}{\cancel{0}}00\rangle_{123} \overset{-}{\cancel{+}} |\overset{0}{\cancel{1}}11\rangle_{123}\right)\left(|000\rangle_{456} \overset{-}{\cancel{+}} |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$b|1_L\rangle = \left(|\overset{1}{\cancel{0}}00\rangle_{123} \overset{+}{\cancel{-}} |\overset{0}{\cancel{1}}11\rangle_{123}\right)\left(|000\rangle_{456} \overset{+}{\cancel{-}} |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

✓ A) $Z_1 Z_2$

✗ B) $Z_4 Z_8$

✓ C) $Z_1 Z_2 Z_3$

✓ D) $X_1 Z_2$

✗ E) $X_1 X_2$ lowest wt correction is X_3

✓ F) $X_1 X_4$

✗ G) $Y_1 Z_4$ vs. $X_1 Z_7$

Fault Tolerant Quantum Computing

Shor Code: All single qubit errors, 9 qubits, 1 logical qubit

Concatenation:

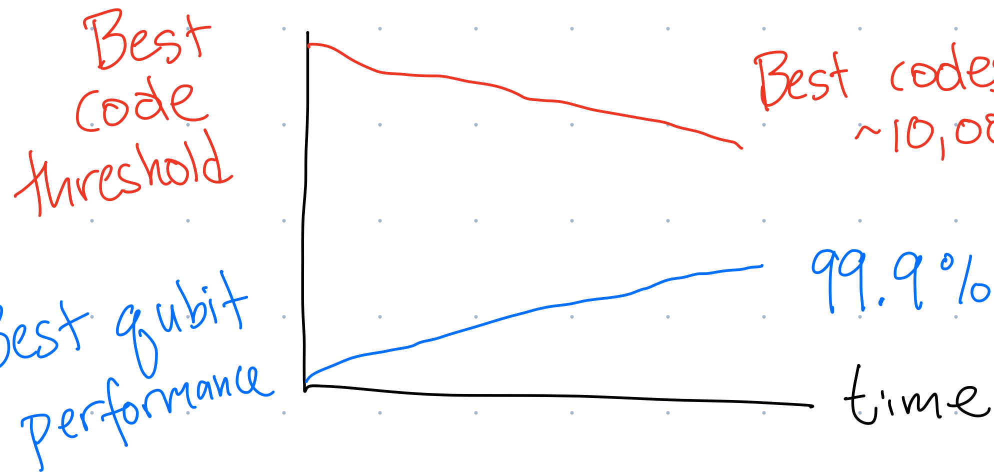
$|0_L\rangle \quad |1_L\rangle$

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

$\Rightarrow$ 81 qubit code that corrects any 2-qubit errors

But... more qubits means more chance of error. Can only keep ahead of accumulating errors if error rate is small enough



Best codes require 99.9% accuracy, ~10,000 physical qubits / logical qubit

99.9% error rate for CNOT, 99.99% for 1 qubit

fluxonium qubit

Use in practice:

Apply error correction every time step

Gates need to be applied in a way that doesn't create more errors.

$$X_L(a|0_L\rangle + b|1_L\rangle) \to a|1_L\rangle + b|0_L\rangle$$

$$X_L(a|000\rangle + b|111\rangle) \to a|111\rangle + b|000\rangle$$

$$X_L = X_1 X_2 X_3$$