ENTANGLEMENT + CHSH

Goals

- Describe entangled states + product states
- Determine if a 2-qubit state is entangled
- Describe why entanglement helps us win CHSH
- Analyze 2-qubit systems

Announcements

- In class exam Thursday QI1, QI2 (15 min, no notes)
- Review where to find notes on Canvas
- OH Today 12:30-1:30

conv: meas in bra

$$|\langle \psi | \phi \rangle| = |\langle \phi | \psi \rangle|$$

Exit Tickets

Exit Tickets

Review CHSH

- Better than 0.85
- $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ vs. $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$
- Randomness $\rightarrow$ PSET
- Need to set up when meet ahead of time

Exit Tickets

- How many qubit states are there?

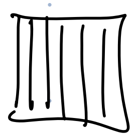
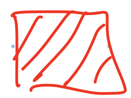
measurement outcome

A) 2 B) 4 $|0\rangle, |1\rangle$ $|+\rangle, |-\rangle$ $|\circlearrowright\rangle$ $|\circlearrowleft\rangle$ C) COUNTABLY ∞ D UNCOUNTABLY ∞

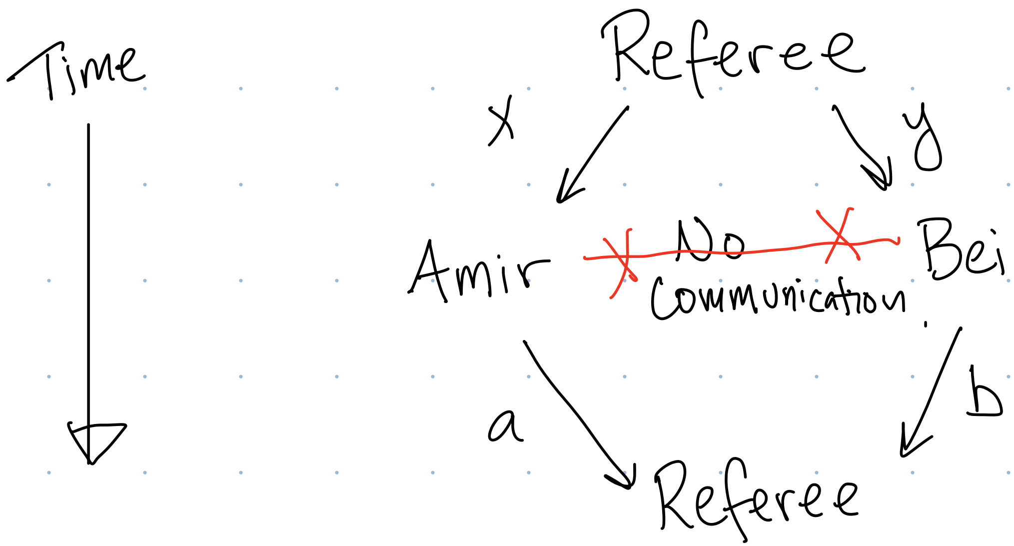
- Does this represent a qubit state?

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle \qquad \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1$$

A) YES B) NO C) Maybe D) IDK

- When $\underline{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}$ 45° vs $\underline{\cos\theta|0\rangle + \sin\theta|1\rangle}$?
- Does orientation/order of $|0\rangle$, $\langle 1|$ matter? $|0\rangle\langle 1|$ $|0\rangle|1\rangle$ $\langle 0|1\rangle$
- Bra physical? Bra name? •  vs $\{|0\rangle, |1\rangle\}$  $\{|+\rangle, |-\rangle\}$
- No sines/cosines on exam (or will provide)

CHSH Game (Clauser, Horne, Shimony, Holt) (Bell Inequality)



$a, b, x, y \in \{0, 1\}$

Amir + Bei win:

x, y	winning condition
00, 01, 10	$a = b$
11	$a \neq b$

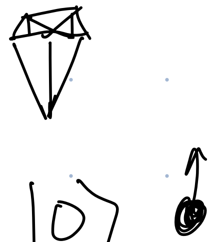
PSet: show w/o quantum best is 75% win rate

What if we give them each a qubit? $|0\rangle$ $|1\rangle$

Diamond nitrogen vacancy qubit

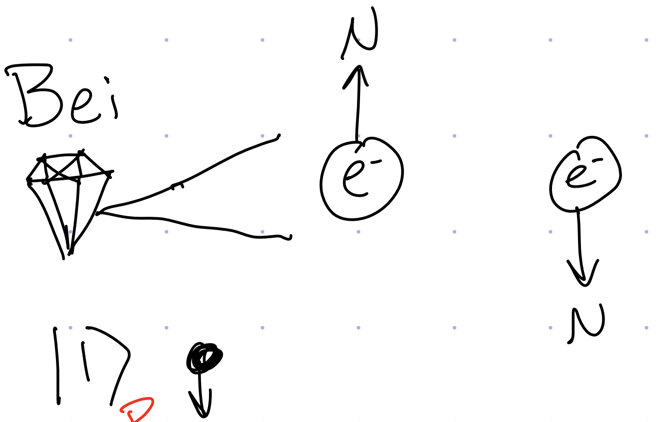
~ 1 second coherence

Amir



$|0\rangle_A$

Bei



$|1\rangle_B$

How do we represent the combined state of both qubits?

✓ A: $|0\rangle \otimes |1\rangle$ ← Tensor Product

✓ B: $|0\rangle_A |1\rangle_B$ ← implied tensor product

✓ C: $\overset{A\,B}{|0\,1\rangle}_{AB}$ ← combine into 1 ket

✓ D: $\overset{A\,B}{|0\,1\rangle}$

1^{st} qubit "A" qubit is $|0\rangle$

inner product

$\langle 0| \to \leftarrow |1\rangle \rightsquigarrow \langle 0|1\rangle \to 0$ (number)

$|0\rangle \to \leftarrow |1\rangle \Rightarrow |01\rangle$

tensor product Bigger kets

"Standard basis states"

Generic 2 Qubit State:

$$|\psi\rangle_{AB} = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

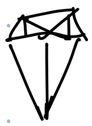
$$\sum_{i \in \{0,1\}^2} |a_i|^2 = 1$$

amplitudes

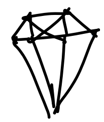
$a_i \in \mathbb{C}$

Strategy:

Amir



Bei

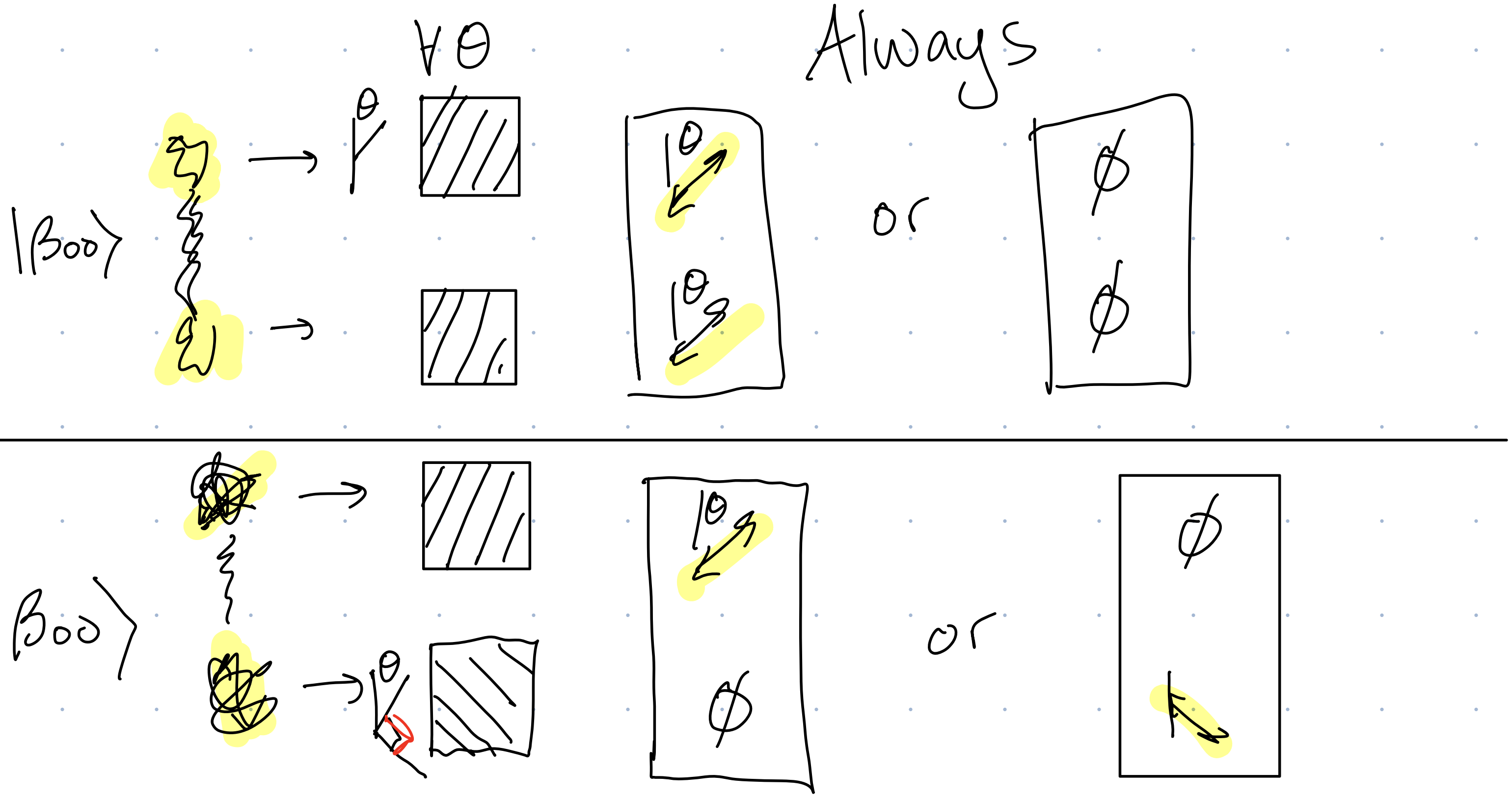


$a_{00} = a_{11} = \frac{1}{\sqrt{2}}$

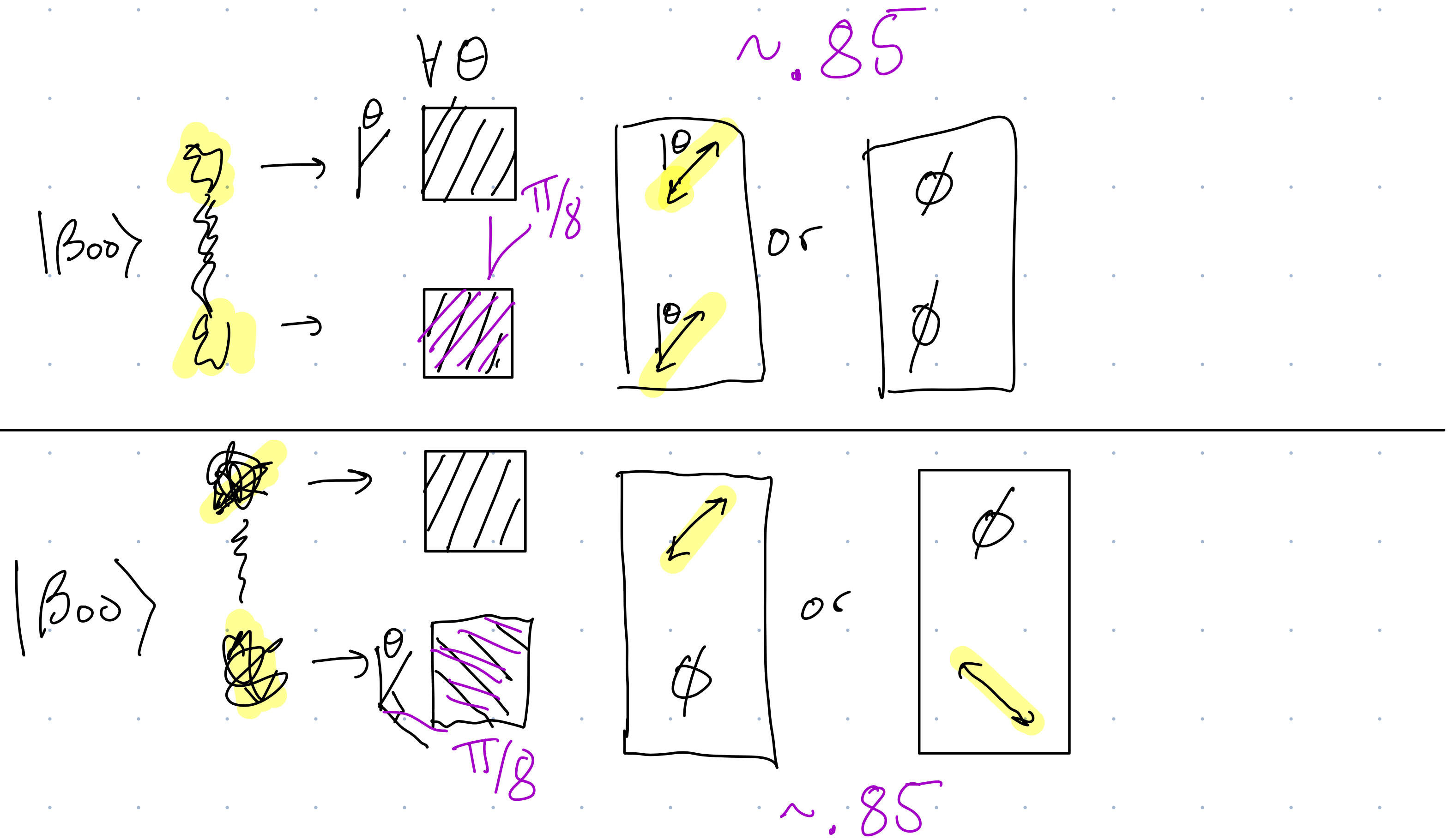
$a_{01} = a_{10} = 0$

$$|\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle_{AB} + \frac{1}{\sqrt{2}}|11\rangle_{AB}$$

$\frac{1}{\sqrt{2}}|00\rangle_{AB} + \frac{1}{\sqrt{2}}|11\rangle_{AB}$ is Entangled!



$\frac{1}{\sqrt{2}}|00\rangle_{AB} + \frac{1}{\sqrt{2}}|11\rangle_{AB}$ is <u>Entangled!</u>



CHSH Game

Referee

$x \swarrow$ $\searrow y$

Amir → ~~X~~ No comms ~~X~~ ← Bei

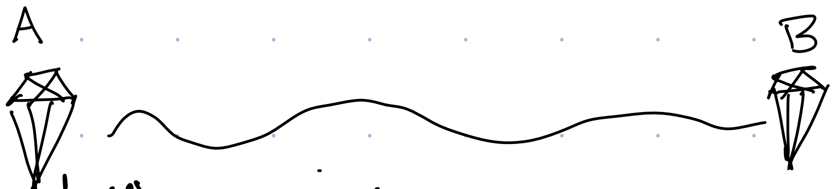
$a \searrow$ $\swarrow b$

Referee

Amir + Bei win

x, y	if
00, 01, 10	$a = b$
11	$a \neq b$

Strategy

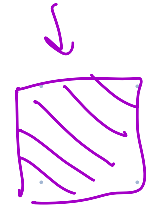


$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

Success Prob? Why? ~85%

Amir Receives	Amir does measurement
$x = 0$	0: ↗ ↕ $a = 0$; ↘ ϕ $a = 1$
$x = 1$	$\pi/4$: ↗ ⤢ $a = 0$; ↘ ϕ $a = 1$

Bei Receives	Bei does measurement
$y = 0$	$\pi/8$: ↗ ⤢ $b = 0$; ↘ ϕ $b = 1$
$y = 1$	$-\pi/8$: ↗ ⤡ $b = 0$; ↘ ϕ $b = 1$

$-\frac{\pi}{4}$ 

Quantum Entanglement

def:

A state $|\psi\rangle_{AB}$ is a ==product state== if $\exists\ |\varphi_1\rangle, |\varphi_2\rangle$ s.t. $|\psi\rangle_{AB} = |\varphi_1\rangle_A |\varphi_2\rangle_B$

A state $|\psi\rangle_{AB}$ is ==entangled== if $\nexists\ |\varphi_1\rangle, |\varphi_2\rangle$ s.t. $|\psi\rangle_{AB} = |\varphi_1\rangle_A |\varphi_2\rangle_B$

There are valid 2-qubit states that can't be described as A system in a state + B system in a state.

Similar to classical correlation $\Longrightarrow$ $Pr(00) = \frac{1}{2}$

$Pr(11) = \frac{1}{2}$

Qubit A

$|\psi_1\rangle_A = a_0|0\rangle + a_1|1\rangle$

Qubit B

$|\psi_2\rangle_B = b_0|0\rangle + b_1|1\rangle$ $\qquad |0\,1\rangle$

Combined: $|\psi\rangle_{AB} = |\psi_1\rangle_A \otimes |\psi_2\rangle_B = |\psi_1\rangle|\psi_2\rangle = \cancel{|\psi_1\psi_2\rangle}$

mult

$$= \left(a_0|0\rangle + a_1|1\rangle\right)_A \left(b_0|0\rangle + b_1|1\rangle\right)_B$$

$$= a_0 b_0 |0\rangle_A|0\rangle_B + a_0 b_1 |0\rangle_A|1\rangle_B + a_1 b_0 |1\rangle_A|0\rangle_B + a_1 b_1 |1\rangle|1\rangle$$

$$= a_0 b_0 |00\rangle_{AB} + a_0 b_1 |01\rangle_{AB} + a_1 b_0 |10\rangle_B + a_1 b_1 |11\rangle$$

Let $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ QI 3

Prove $|\beta_{00}\rangle$ is entangled.

$a|0\rangle + b|1\rangle$

Pf: Assume for contradiction $|\beta_{00}\rangle$ is not entangled.

there exists such that

$$\boxed{\text{A state } |\psi\rangle_{AB} \text{ is not entangled if } \exists\, |\psi_1\rangle, |\psi_2\rangle \text{ s.t. } |\psi\rangle_{AB} = |\psi_1\rangle_A |\psi_2\rangle_B}$$

$c|0\rangle + d|1\rangle$

$$(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle)$$

0 $\qquad$ 0

$$\boxed{ac}|00\rangle + bc|10\rangle + ad|01\rangle + \boxed{bd}|11\rangle = \boxed{\frac{1}{\sqrt{2}}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Contra. $\qquad$ Contradiction

$bc = 0 \Rightarrow b = 0$

or

$c = 0$

① If have 2-qubit state $|\psi\rangle_{AB}$ and measure A qubit with $M_A = \{|\alpha_0\rangle, |\alpha_1\rangle\}$ and B qubit with
$M_B = \{|\beta_0\rangle, |\beta_1\rangle\}$

③ Get outcome $|\alpha_i\rangle_A|\beta_j\rangle_B$ with prob. $|\langle\alpha_i|_A\langle\beta_j|_B|\psi\rangle_{AB}|^2$

$$\left|\langle\psi|_{AB}|\alpha_i\rangle_A|\beta_j\rangle_B\right|^2$$

$$\shortparallel$$

Always keep AB order

state collapses to

$$|\alpha_i\rangle_A|\beta_j\rangle_B$$

② How many measurement outcomes are there?

1 A) 2 (M_A and M_B)

2 B) 4 ($|\alpha_0\rangle|\beta_0\rangle$, $|\alpha_0\rangle|\beta_1\rangle$, ...)

3 C) Not enough information

① If have 2-qubit state $|\psi\rangle_{AB}$ and measure A qubit with $M_A = \{|\alpha_0\rangle, |\alpha_1\rangle\}$ and B qubit with
$M_B = \{|\beta_0\rangle, |\beta_1\rangle\}$

$$\left|\langle\psi|_{AB}\, |\alpha_i\rangle_A\, |\beta_j\rangle_B\right|^2$$

$$\shortparallel$$

③ Get outcome $|\alpha_i\rangle_A |\beta_j\rangle_B$ with prob. $\left|\langle\alpha_i|_A \langle\beta_j|_B\, |\psi\rangle_{AB}\right|^2$,
state collapses to $|\alpha_i\rangle_A |\beta_j\rangle_B$

④ Suppose Amir + Bei cannot communicate, but share $|\psi\rangle_{AB}$ (unknown state). If Amir measures

$M_A = \{|0\rangle, |1\rangle\}$ and Bei measures $M_B = \{|+\rangle, |-\rangle\}$ and they get outcome $|0\rangle_A |+\rangle_B$, what does Amir know about the two qubits after the measurements if no communication?

① A) Nothing

② B) A is in $|0\rangle$, B unknown

③ C) A is in $|0\rangle$, B in $|+\rangle$

Example: $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$$\boxed{\left|\langle\alpha_i|_A \langle\beta_j|_B |\psi\rangle_{AB}\right|^2}$$

$M_A = \{|+\rangle, |-\rangle\}$ $M_B = \{|+\rangle, |-\rangle\}$

Probability of $|+\rangle_A|-\rangle_B$?

tensor product

inner products

$$\langle +|_A \langle -|_B |\psi\rangle_{AB} = \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}\langle 0| - \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \left(\frac{1}{2}\langle 00| - \frac{1}{2}\langle 01| + \frac{1}{2}\langle 10| - \frac{1}{2}\langle 11|\right)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\langle 00|00\rangle^{1} + \cancel{\frac{1}{2\sqrt{2}}\langle 00|11\rangle}^{0} + \ldots$$

Rule $\langle x|y\rangle = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{else} \end{cases}$ where x, y are bit strings / standard basis states

$$= \frac{1}{2\sqrt{2}}\langle 00|00\rangle^{1} - \frac{1}{2\sqrt{2}}\langle 11|11\rangle^{1} = 0 \Rightarrow |0|^2 = 0$$

Practice: Start PS3 #4a

$$\left(\langle \phi_0(\xi, x) |_A \langle \phi_0(\xi, x) |_B \right) \left(\frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle \right)$$

$$e^{ix} \Rightarrow e^{-ix}$$