Deutsch's Algorithm

Learning Goals

- Design + analyze a quant. algorithm [QC2]
- Read circuit diagrams
- Describe time + query complexity

Announcements

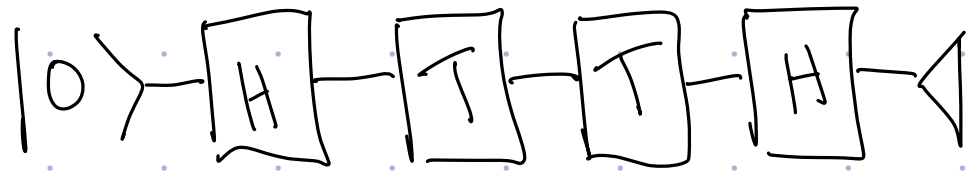
- Exam 2: 10/22, 75 min in class
  QI2-5, QC1
  1 side of 3"x5" notecard
- go/techJam26

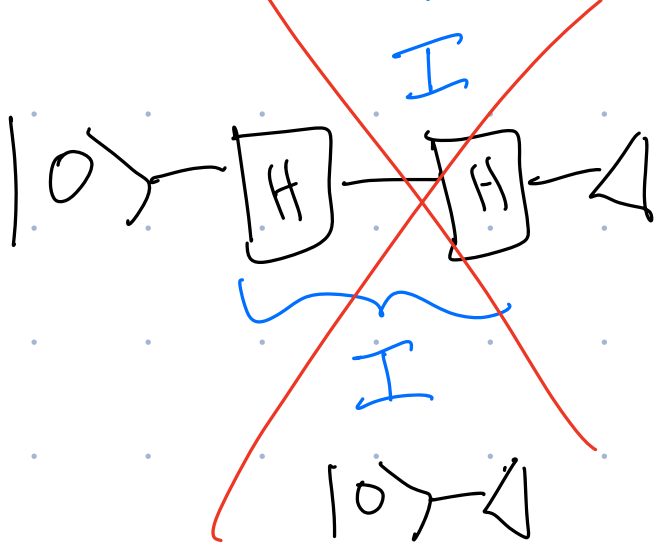
T = I?

ET: f?

Warm-up: QC1:

$T: |0\rangle \to |0\rangle$
$|1\rangle \to e^{i\pi/4}|1\rangle$





$$H|0\rangle \to \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$TH|0\rangle \to \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|1\rangle \qquad e^{i\pi/2}$$

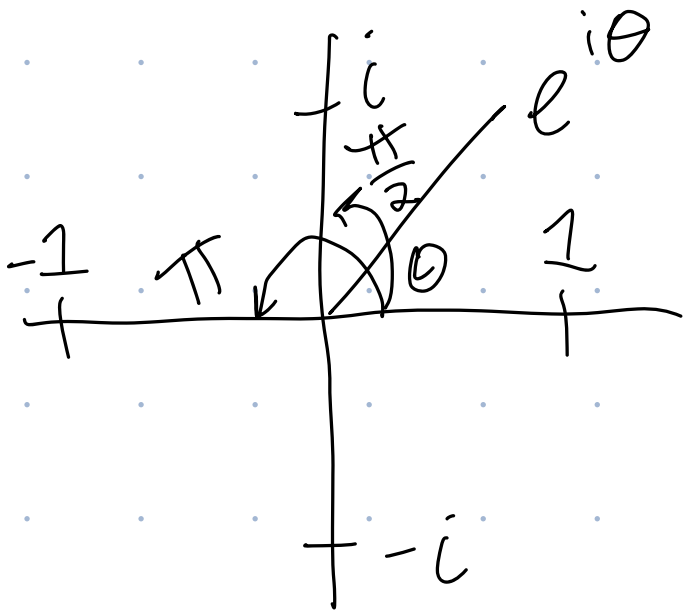
$$TTH|0\rangle \to \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi/4}\cdot e^{i\pi/4}}{\sqrt{2}}|1\rangle$$

$$HTTH|0\rangle \to \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{e^{i\pi/2}}{2}|0\rangle - \frac{e^{i\pi/2}}{2}|1\rangle$$

$$|\psi\rangle = \frac{1-e^{i\pi/2}}{2}|1\rangle + \text{—}|0\rangle$$

$$|\langle 1|\psi\rangle|^2 = \left|\frac{1-e^{i\pi/2}}{2}\right|^2 = \left|\frac{1-i}{2}\right|^2 = \frac{1}{2}$$

$$\left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$



Deutsch's algorithm analyzes a 1-bit function f:

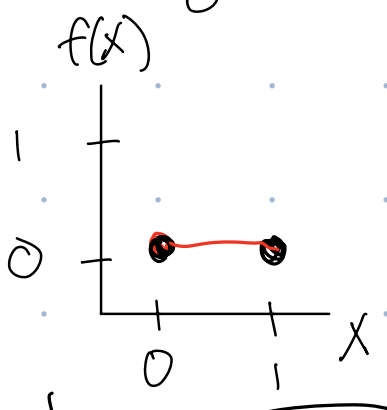
x	f(x)
0	f(0)
1	f(1)

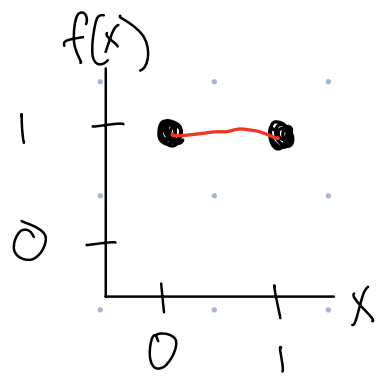
$\leftarrow \in \{0,1\}$

ex: $f(x) = \begin{cases} 1 & \text{if it will rain at time } x \\ 0 & \text{if it will not rain at time } x \end{cases}$

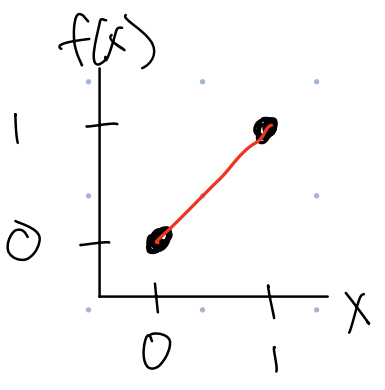
$x = \begin{cases} 0 \rightarrow \text{day} \\ 1 \rightarrow \text{night} \end{cases}$

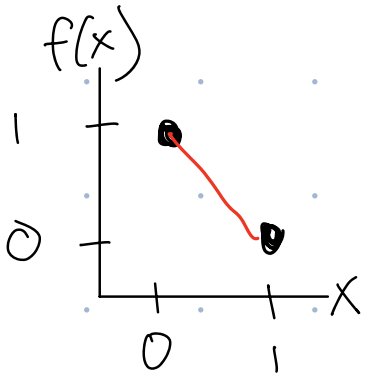
Only 4 1-bit Functions:





flat (even)



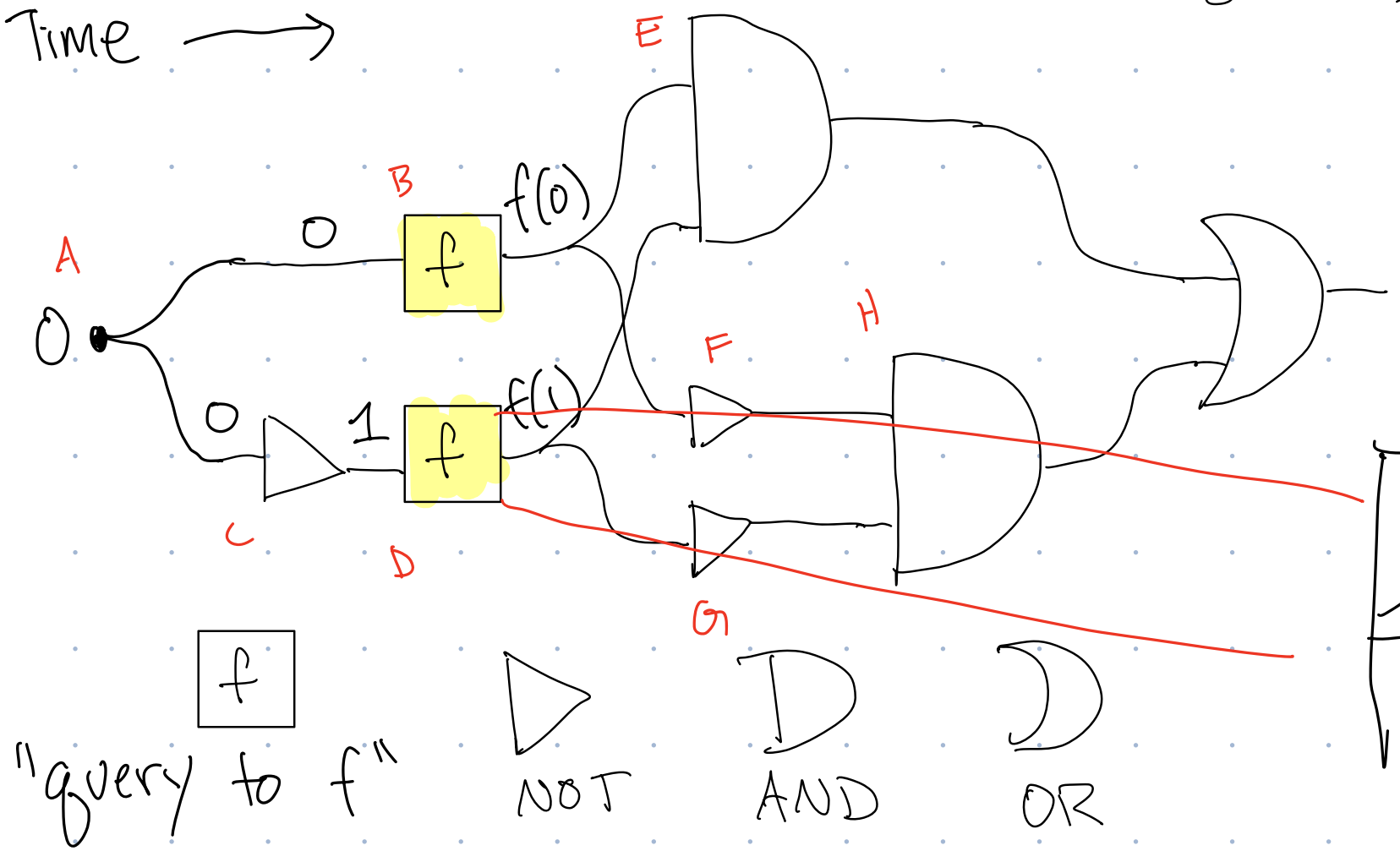


sloped (balanced)

Deutsch's Problem: Determine if f is flat or sloped.

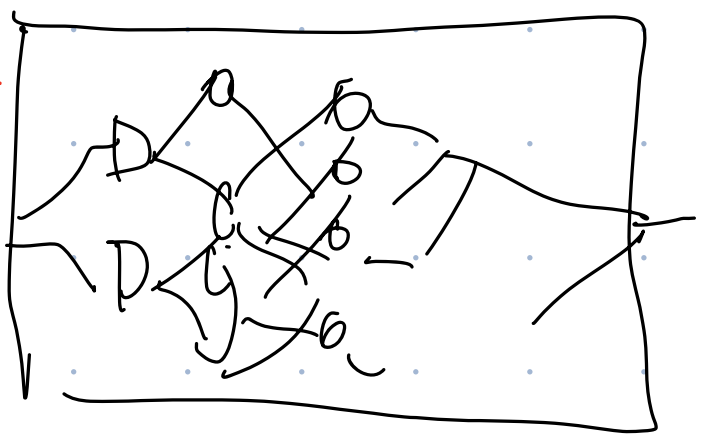
A Classical Algorithm (circuit diagram)

- wires are bits
- shapes are gates



1 ⇒ flat

0 ⇒ sloped



↳ time to run f

Query Complexity = 2 Time Complexity = 5 steps
8 steps

We would like to determine if f is flat or sloped using as few queries as possible.

What is the minimum number of classical queries to f needed to determine flat/sloped

A) 0 B) 1 C) 2

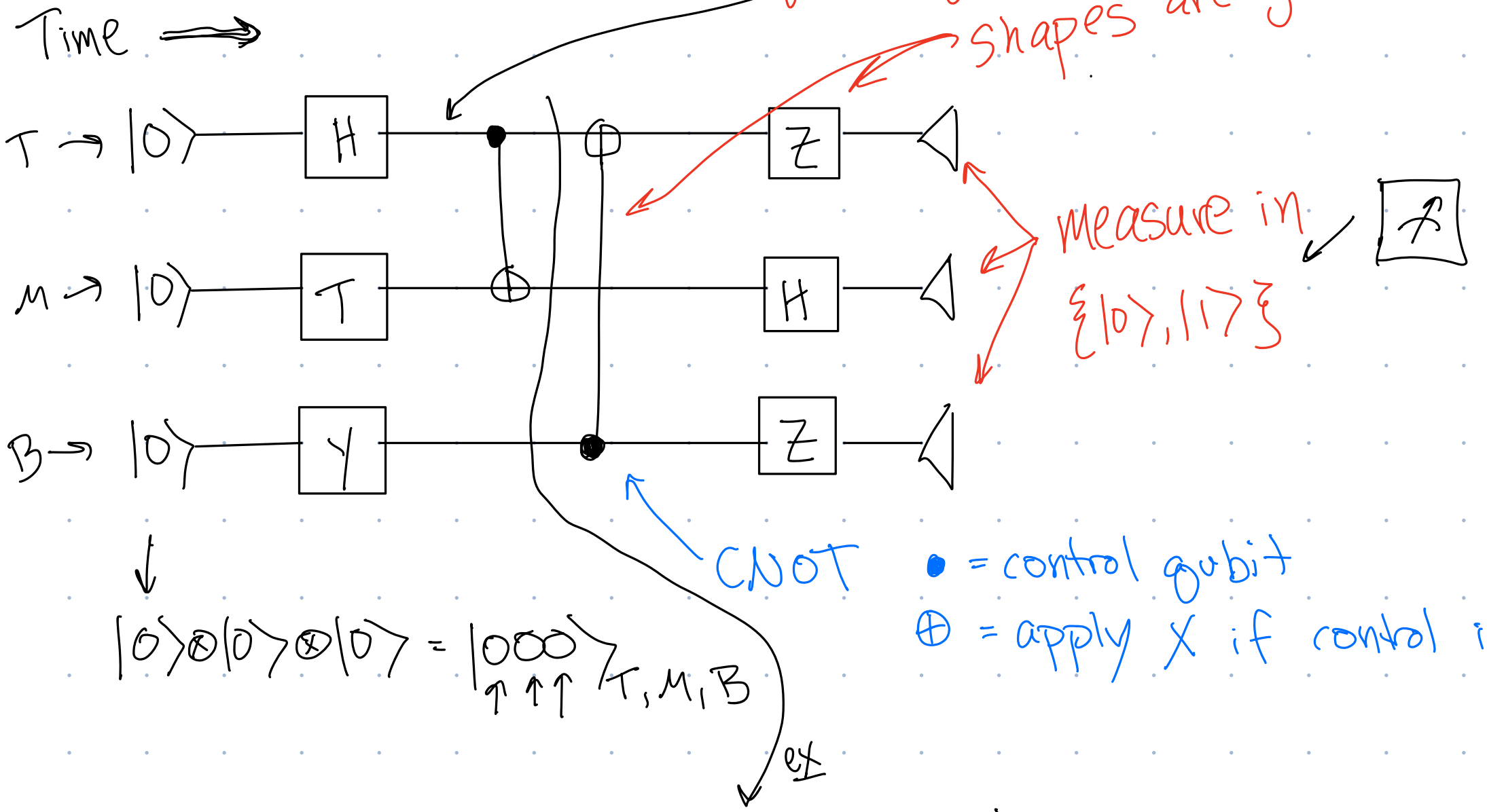
Deutsch's Problem

- Classical Query Complexity: 2
- Quantum Query Complexity: 1

Why do we care?!

- Shows quantum advantage
- Techniques that used bigger alg.

Quantum Circuit



$|0\rangle \otimes |0\rangle \otimes |0\rangle = |000\rangle_{T,M,B}$

ex

$$CNOT_{3\to 1}\left(\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|001\rangle\right) \to \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|101\rangle$$

Need gate for f. What about:

$$f(0) = f(1) = 0$$

$U_f|0\rangle \to |f(0)\rangle$

$U_f|1\rangle \to |f(1)\rangle$

$U_f|0\rangle = |0\rangle$ $U_f|1\rangle = |0\rangle$ $\xrightarrow{U_f}$ $|\psi\rangle$ $|\psi\rangle$

CNOT
X
AND, NOT

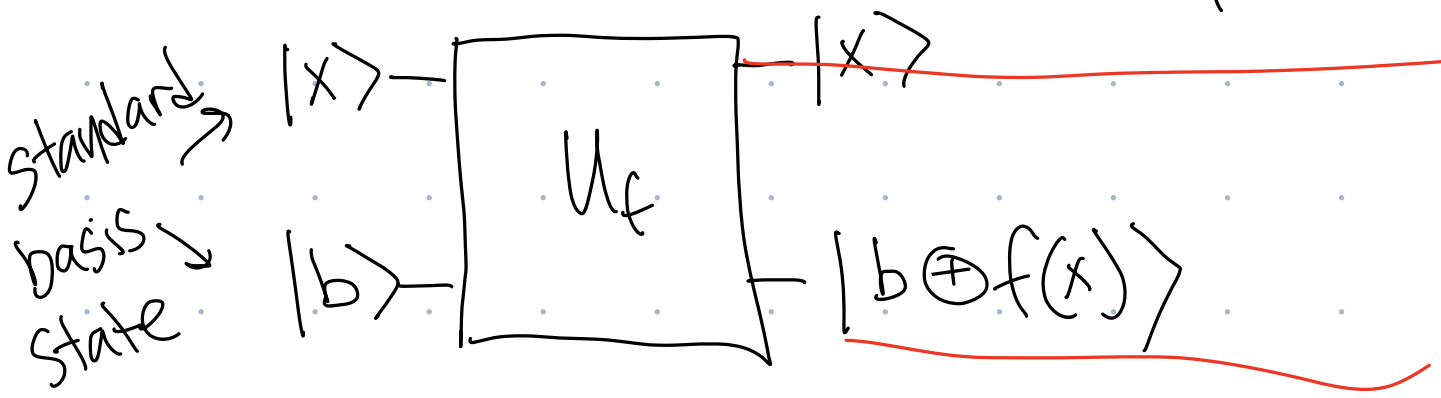
Explain why this is not an allowed gate.

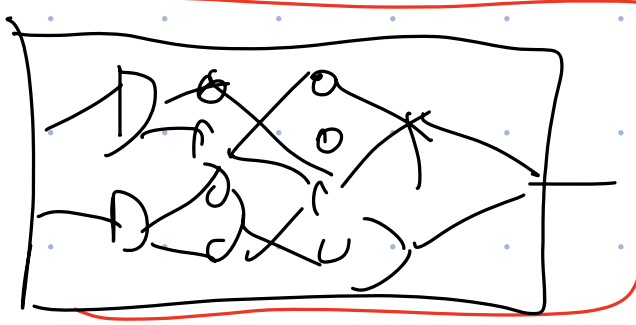
* State → State
* Reversible ✗

$U_f|00\rangle = |0, 0 \oplus f(0)\rangle = |0, f(0)\rangle$

$U_f|10\rangle = |1, 0 \oplus f(1)\rangle = |1, f(1)\rangle$

Instead:



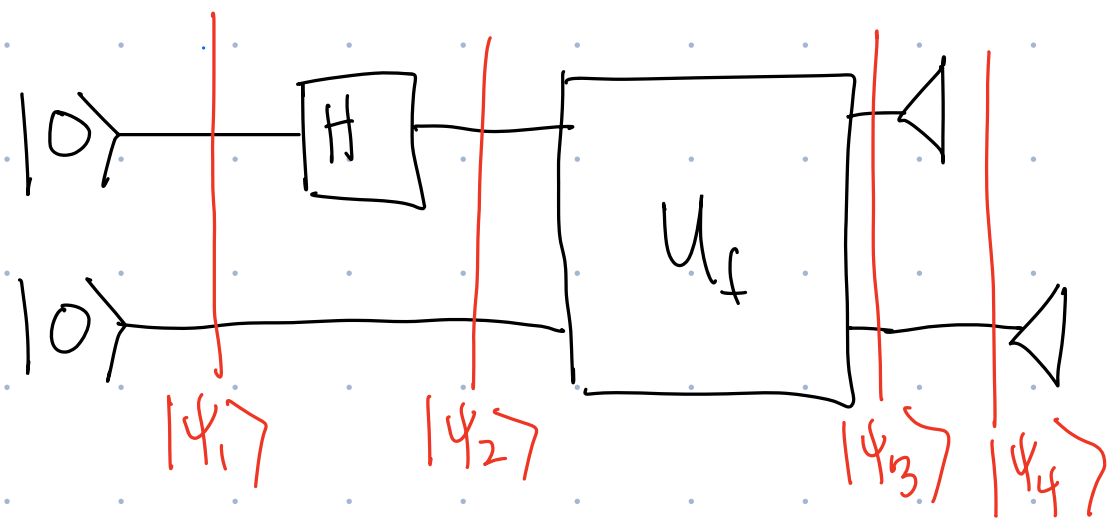


$U_f U_f = I$

$$U_f|x\rangle|b\rangle = |x\rangle|b \oplus f(x)\rangle$$

* $x, b \in \{0, 1\}$

Analyzing a Quantum Circuit



$$U_f|x\rangle|b\rangle = |x\rangle|b \oplus f(x)\rangle$$

* $x, b \in \{0,1\}$

$$|\psi_1\rangle = |0\rangle \otimes |0\rangle = |00\rangle$$

$$|\psi_2\rangle = (H|0\rangle) \otimes |0\rangle = |+\rangle|0\rangle$$

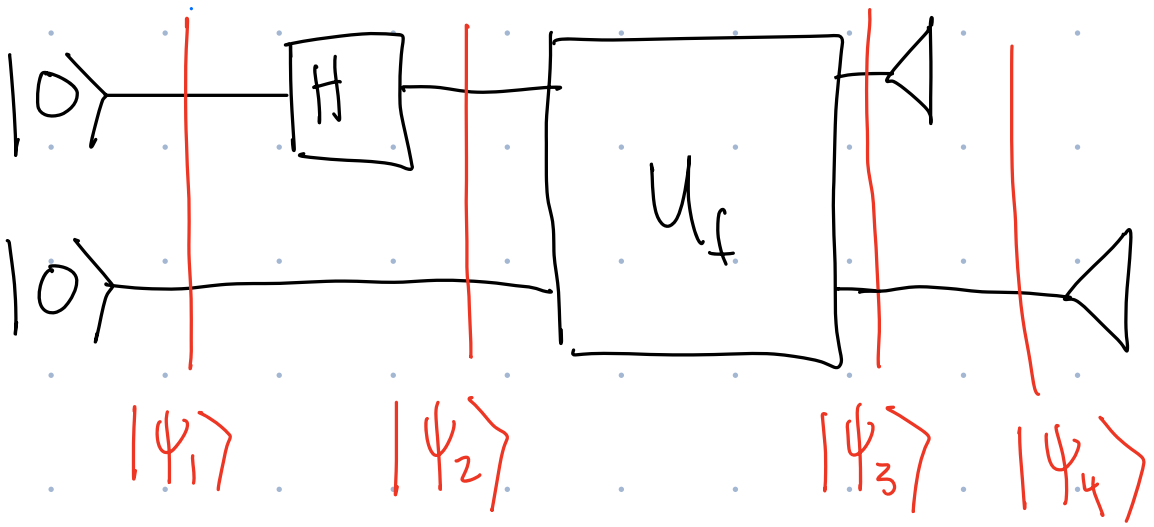
$$|\psi_3\rangle = U_f|+\rangle|0\rangle = U_f\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)|0\rangle = \tfrac{1}{\sqrt{2}}U_f|0\rangle|0\rangle + \tfrac{1}{\sqrt{2}}U_f|1\rangle|0\rangle$$

$$= \tfrac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|f(1)\rangle = |0\rangle\left(\tfrac{1}{\sqrt{2}}|f(0)\rangle\right) + |1\rangle\left(\tfrac{1}{\sqrt{2}}|f(1)\rangle\right)$$

Partial Measurement of Top Qubit:

Outcome $|0\rangle \rightarrow$ Prob $\left|\tfrac{1}{\sqrt{2}}\right|^2 \Rightarrow |0\rangle|f(0)\rangle$

Outcome $|1\rangle \rightarrow$ Prob $\left|\tfrac{1}{\sqrt{2}}\right|^2 \Rightarrow |1\rangle|f(1)\rangle$



Partial Measurement:

Outcome $|0\rangle$

Outcome $|1\rangle$

Mini-Hype Lesson

You need more than superposition to get a quantum advantage.

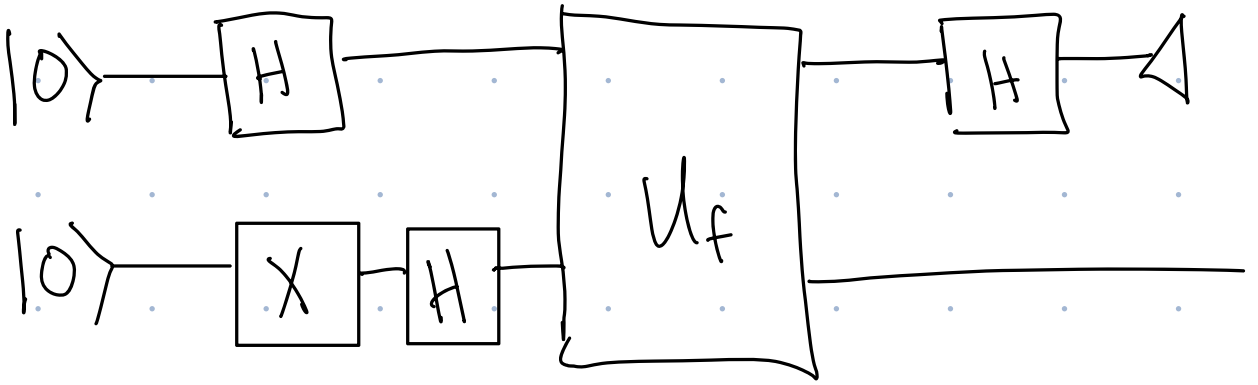
New rule for U_f's behavior:

New Rule

$x \in \{0,1\}$

$$U_f |x\rangle|-\rangle = (-1)^{f(x)} |x\rangle|-\rangle$$

① Analyze outcome of this circuit using this rule

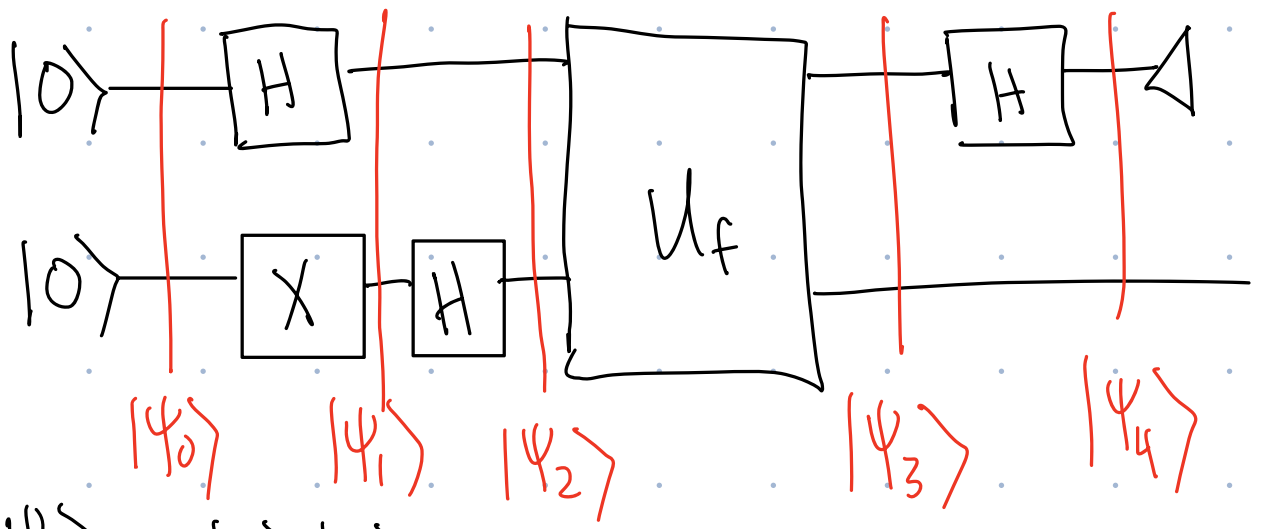


② Use old rule to derive new rule

Old Rule

$x, b \in \{0,1\}$:

$$U_f |x\rangle|b\rangle = |x\rangle|b \oplus f(x)\rangle$$



$$|\psi_0\rangle = |0\rangle|0\rangle$$

$$|\psi_1\rangle = (H \otimes X)|0\rangle|0\rangle = |+\rangle|1\rangle$$

$$|\psi_2\rangle = (I \otimes H)|+\rangle|1\rangle = |+\rangle|-\rangle$$

$$|\psi_3\rangle = U_f|+\rangle|-\rangle = U_f\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|-\rangle$$

$$= \frac{1}{\sqrt{2}} U_f|0\rangle|-\rangle + \frac{1}{\sqrt{2}} U_f|1\rangle|-\rangle$$

$$= \frac{1}{\sqrt{2}} (-1)^{f(0)}|0\rangle|-\rangle + \frac{1}{\sqrt{2}} (-1)^{f(1)}|1\rangle|-\rangle$$

$$= \frac{1}{\sqrt{2}} \left((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle \right) \otimes |-\rangle$$

Ignore

$$H\left(\frac{1}{\sqrt{2}}(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right)$$

$$\frac{1}{\sqrt{2}}(-1)^{f(0)}|+\rangle + \frac{1}{\sqrt{2}}(-1)^{f(1)}|-\rangle$$

$$= \frac{1}{\sqrt{2}}(-1)^{f(0)}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(-1)^{f(1)}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{2}\left((-1)^{f(0)} + (-1)^{f(1)}\right)|0\rangle + \frac{1}{2}\left((-1)^{f(0)} - (-1)^{f(1)}\right)|1\rangle$$

$$Pr(|0\rangle) = \left|\frac{1}{2}\left((-1)^{f(0)} + (-1)^{f(1)}\right)\right|^2$$

$f(0)$: 1, $f(1)$: 1 → -1, -1

If sloped, $Pr(|0\rangle) = 0$ If flat, $Pr(|0\rangle) = 1$

Phase Kickback

$$U_f : |x\rangle|-\rangle \rightarrow (-1)^{f(x)}|x\rangle|-\rangle$$

Info about f's value is stored in phase
Rather than in standard basis.

Phase Kickback

