

Learning Goals

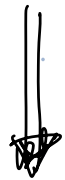
- Design + analyze a quant. algorithm
- Read circuit diagrams
- Describe time + query complexity

Announcements

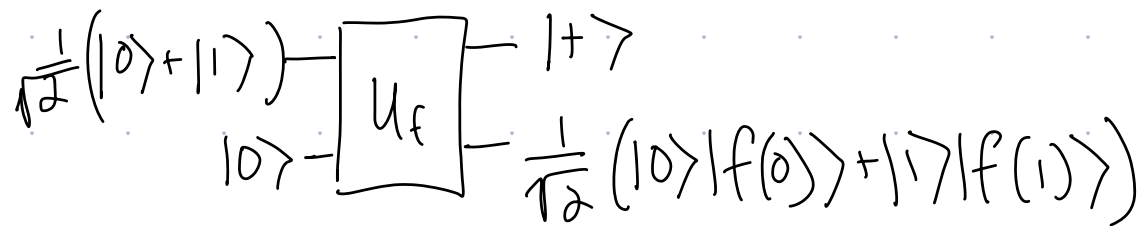
- In-class long exam (2 weeks)
 - Q12, Q13, QC1 (All foundational)
 - Q14, QC2 (interm.)
 - Q15 (advanced)

cheatsheet
3" x 5" notecard

Focus



Exit Tickets



If you are applying U_f to a superposition state, shouldn't this count as 2 queries? One for each part of the state?

What is $f(x)$?

Distributing kets: Keep order!

"Unitary" = "gate"

$$|0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|01\rangle$$

U_A Nothing B

$U_A \otimes I_B$

- Gates/Unitaries to left of state/ket:

$$U |\psi\rangle = |\psi'\rangle$$

- Multiple gates:

$$U_3 U_2 U_1 |\psi\rangle$$

1st applied

2nd applied

What is query complexity?

What is time complexity?

Deutsch's algorithm analyzes a 1-bit function f :

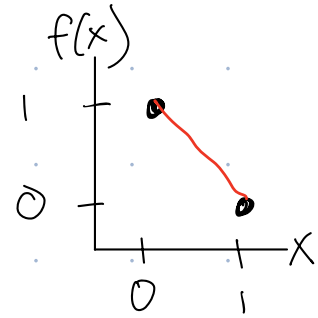
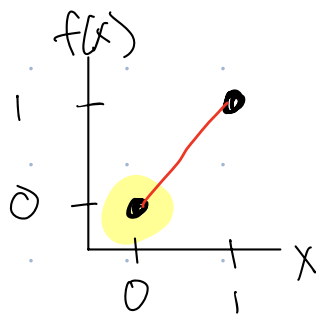
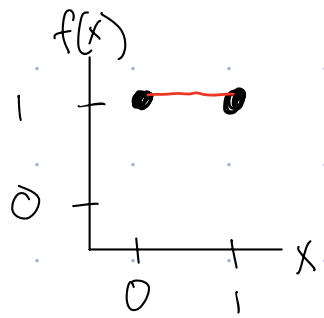
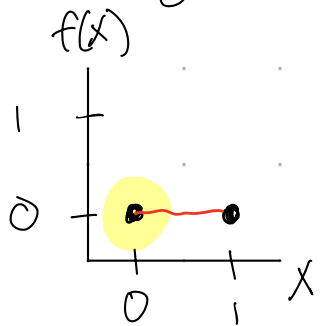
x	$f(x)$
0	$f(0)$
1	$f(1)$

$$f(0), f(1) \in \{0, 1\}$$

ex: $f(x) = \begin{cases} 1 & \text{if it will rain at time } x \\ 0 & \text{if it will not rain at time } x \end{cases}$

$x = \begin{cases} 0 & \rightarrow \text{day} \\ 1 & \rightarrow \text{night} \end{cases}$

Only 4 1-bit Functions:

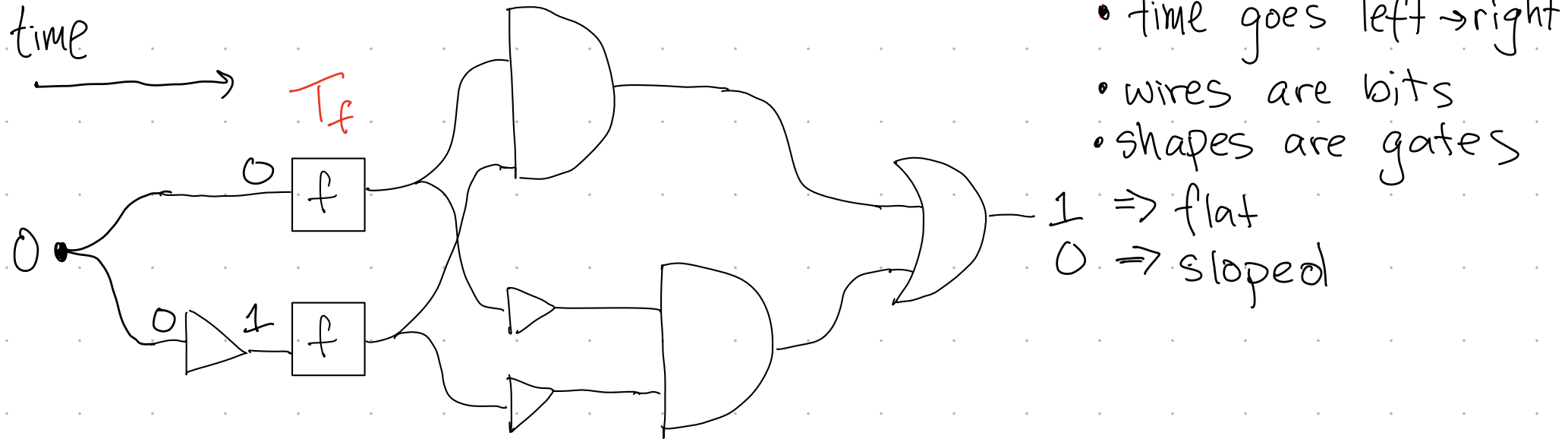


flat (even)

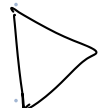
sloped (balanced)

Deutsch's Problem: Given query access to f , determine if flat or sloped

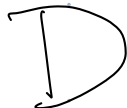
A Classical Algorithm (circuit diagram)



"query to f "



NOT



AND



OR

$\rightarrow$ # of times the gate f is used

Query Complexity = 2

Time Complexity = time to run
 $2T_f + 6$ // $T_f + 3$

We would like to determine if f is flat or sloped using as few queries as possible.

What is the minimum number of classical queries to f needed to determine flat/sloped

A) 0

B) 1

C) 2

Deutsch's Problem

- Classical Query Complexity: 2
- Quantum Query Complexity: 1

Why do we care?!

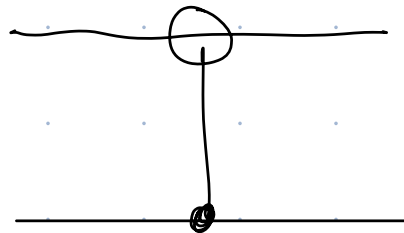
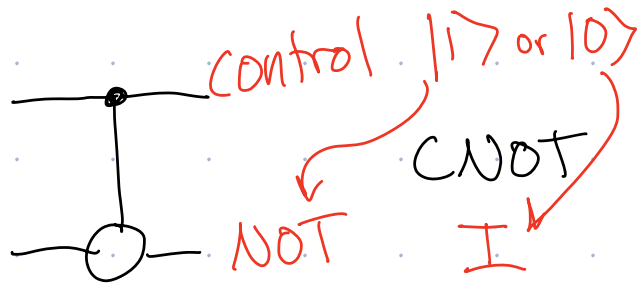
Quantum Gates (Circuit picture)

Time $\rightarrow$

$|0\rangle$

Initialize each qubit
 $|0\rangle$ (usually)

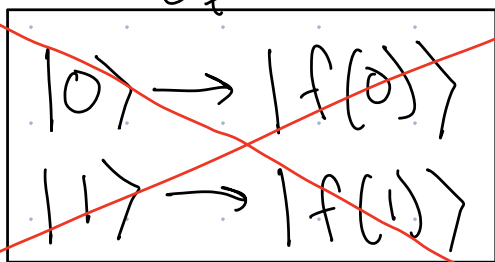
- time $L \rightarrow R$
- wires are qubits
- Gates + measurements are shapes



Measure in standard basis
 $M = \{ |0\rangle, |1\rangle \}$

Need gate for f . What about:

$$f(0) = f(1) = 1 \quad (\text{flat})$$



$$|0\rangle \rightarrow |1\rangle$$

$$|1\rangle \rightarrow |1\rangle$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow$$

$$\frac{1}{\sqrt{2}}(|1\rangle + |1\rangle) = \frac{2}{\sqrt{2}}|1\rangle$$

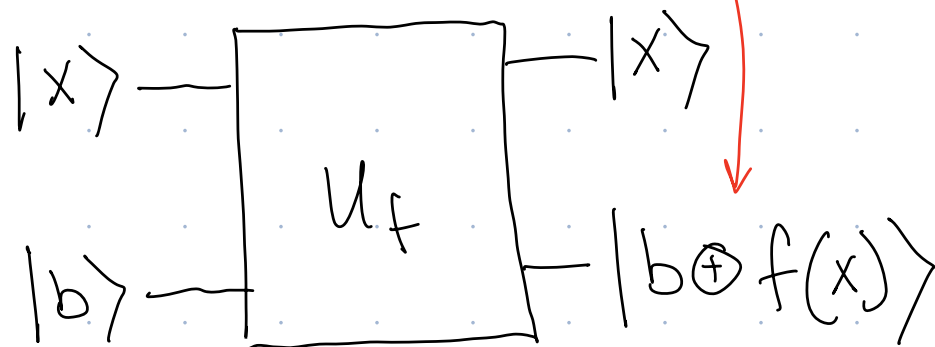
Explain why this is not an allowed gate.

* State $\rightarrow$ State

* Reversible

$$U_f U_f = I$$

Instead:

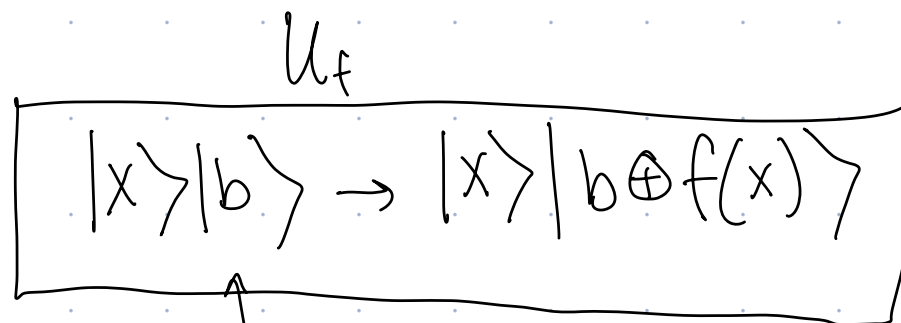


$$x, b \in \{0, 1\}$$

$\downarrow U_f$

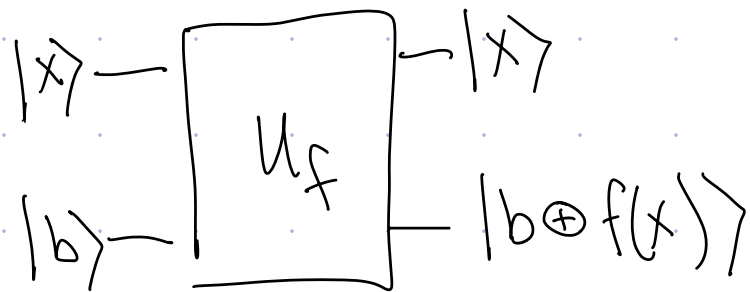
$$|b \oplus f(x) \oplus f(x)\rangle$$

add mod 2



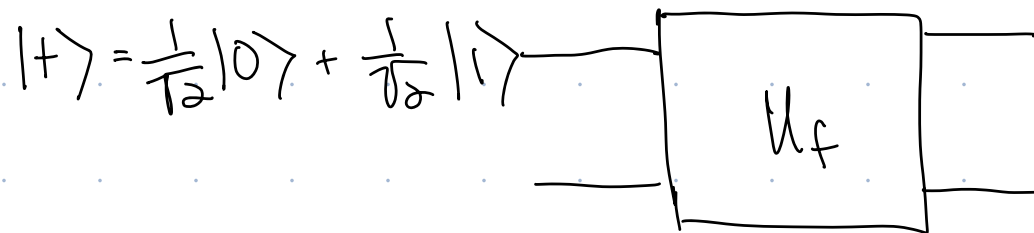
This only works if x, b are bits

$|x\rangle, |b\rangle$ are standard basis states

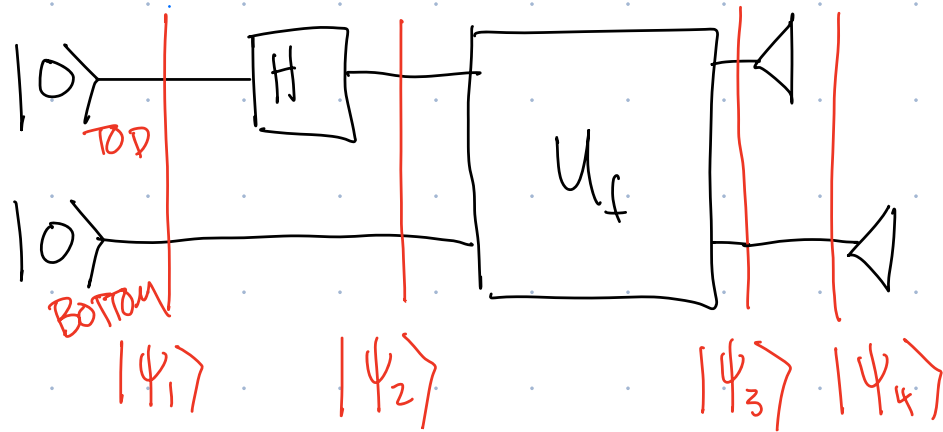


If $x=0$ get info about $f(0)$
 If $x=1$ get info about $f(1)$

Idea - **Superposition!**



Analyzing a Quantum Circuit



$$U_f$$

$$|x\rangle|b\rangle \rightarrow |x\rangle|b \oplus f(x)\rangle$$

$$|0\rangle|0\rangle \rightarrow |0\rangle|0 \oplus f(0)\rangle$$

$$= |0\rangle|f(0)\rangle$$

$$|1\rangle|0\rangle \rightarrow |1\rangle|0 \oplus f(1)\rangle$$

$$= |1\rangle|f(1)\rangle$$

$$|\psi_1\rangle = |0\rangle|0\rangle$$

TOP BOTTOM

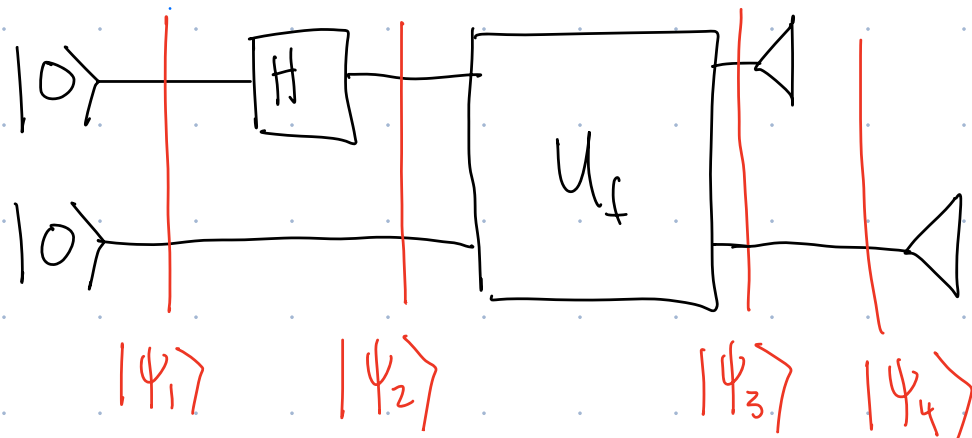
$$|\psi_2\rangle = (H \otimes I)|0\rangle|0\rangle = |+\rangle|0\rangle$$

$$|\psi_3\rangle = U_f(|+\rangle|0\rangle) = |+\rangle|0 \oplus f(+)\rangle$$

$$= U_f\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle = U_f\left(\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle$$

=



$$|\psi_3\rangle = \frac{1}{\sqrt{2}} |0\rangle |f(0)\rangle + \frac{1}{\sqrt{2}} |1\rangle |f(1)\rangle$$

$$= |0\rangle \left(\frac{1}{\sqrt{2}} |f(0)\rangle \right) + |1\rangle \left(\frac{1}{\sqrt{2}} |f(1)\rangle \right)$$

Partial Measurement:

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Outcome $|0\rangle$

- Prob $\left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$

- Collapse $\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} |f(0)\rangle \right) = |f(0)\rangle$

Outcome $|1\rangle$

- Prob $\left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$

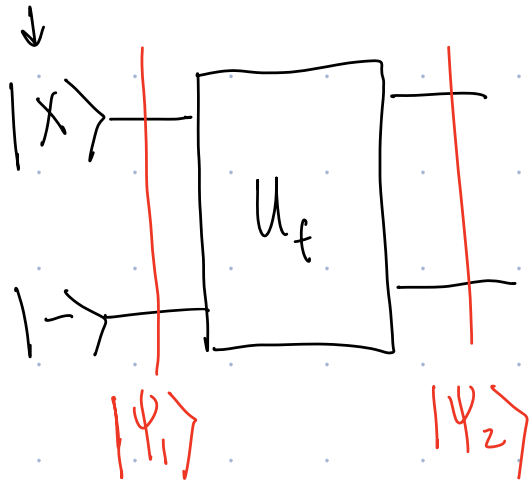
- Collapse $\rightarrow |f(1)\rangle$

Mini-Hype Lesson

You need more than superposition to get a quantum advantage.

Group Exercise:

$x = 0 \text{ or } 1$

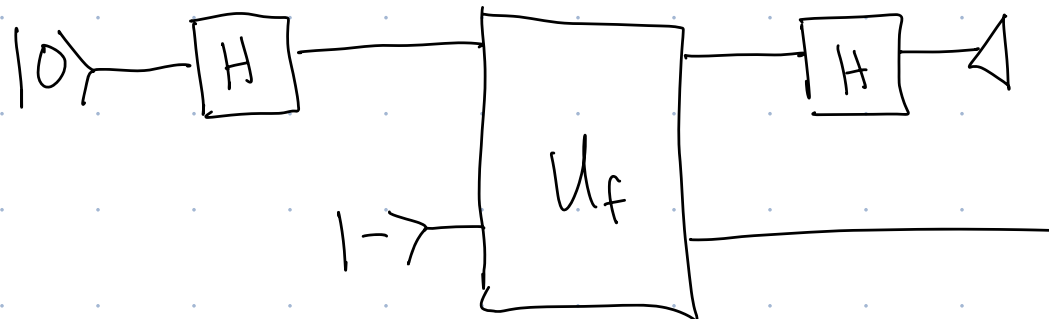


Show that $|\psi_2\rangle = (-1)^{f(x)} |x\rangle |-\rangle$

If $f(x) = 0$

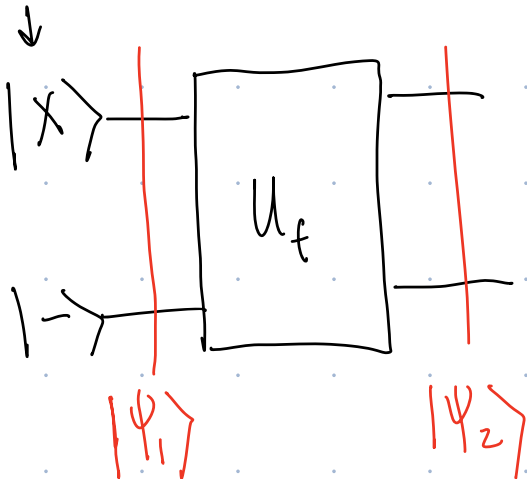
$\downarrow H f(x) = 1$

Analyze outcome of this circuit:



Solution

$x = 0 \text{ or } 1$



$$(-1)^0 = 1 \quad (-1)^1 = -1$$

$$|x\rangle|b\rangle \rightarrow |x\rangle|b \oplus f(x)\rangle$$

Show that $|\psi_2\rangle = (-1)^{f(x)} |x\rangle|-\rangle$

$$\begin{aligned} |\psi_2\rangle &= U_f |x\rangle|-\rangle = U_f |x\rangle \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right) \\ &= U_f \left(\frac{1}{\sqrt{2}} |x\rangle|0\rangle - \frac{1}{\sqrt{2}} |x\rangle|1\rangle \right) \end{aligned}$$

$$= \frac{1}{\sqrt{2}} U_f |x\rangle|0\rangle - \frac{1}{\sqrt{2}} U_f |x\rangle|1\rangle$$

$$= \frac{1}{\sqrt{2}} |x\rangle|0 \oplus f(x)\rangle - \frac{1}{\sqrt{2}} |x\rangle|1 \oplus f(x)\rangle$$

New rule for U_f :

$$|x\rangle|-\rangle = (-1)^{f(x)} |x\rangle|-\rangle$$

$|x\rangle$ standard basis

$$f(x) = 0$$

$$\frac{1}{\sqrt{2}} |x\rangle|0\rangle - \frac{1}{\sqrt{2}} |x\rangle|1\rangle$$

$$|x\rangle \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right)$$

$$= |x\rangle|-\rangle$$

$$f(x) = 1$$

$$\frac{1}{\sqrt{2}} |x\rangle|1\rangle - \frac{1}{\sqrt{2}} |x\rangle|0\rangle$$

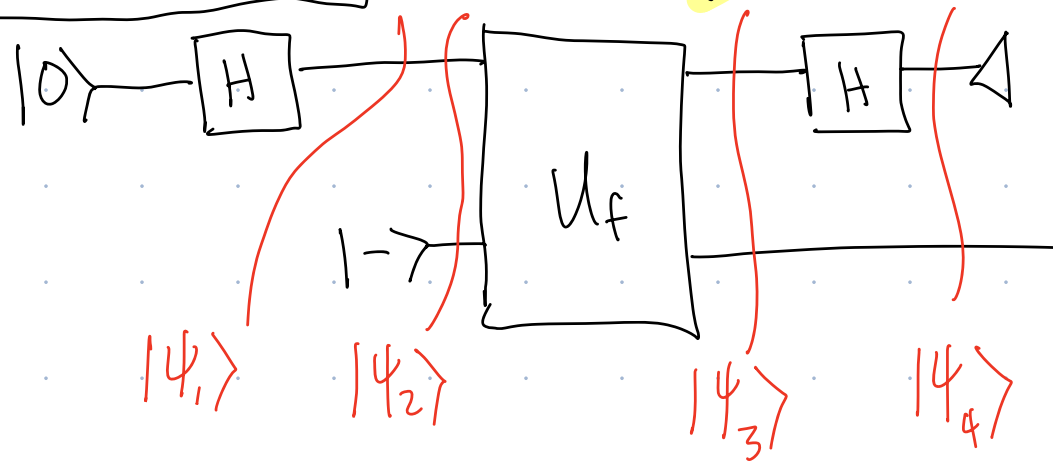
$$|x\rangle \left(-\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right)$$

$$= -|x\rangle|-\rangle$$

Phase Kickback

$$U_f : |x\rangle|-\rangle \rightarrow (-1)^{f(x)}|x\rangle|-\rangle$$

Favorite Class



$$|\psi_1\rangle = H|0\rangle = |+\rangle$$

$$|\psi_2\rangle = |+\rangle|-\rangle$$

$$|\psi_3\rangle = U_f |+\rangle|-\rangle = U_f \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) |-\rangle = U_f \left(\frac{1}{\sqrt{2}}|0\rangle|-\rangle + \frac{1}{\sqrt{2}}|1\rangle|-\rangle \right)$$

$$= \frac{1}{\sqrt{2}} U_f |0\rangle|-\rangle + \frac{1}{\sqrt{2}} U_f |1\rangle|-\rangle = \frac{1}{\sqrt{2}} (-1)^{f(0)} |0\rangle|-\rangle + \frac{1}{\sqrt{2}} (-1)^{f(1)} |1\rangle|-\rangle$$

flat

$$(-1)^{f(0)} = (-1)^{f(1)}$$

$$(-1)^{f(0)} \left(\frac{1}{\sqrt{2}}|0\rangle|-\rangle + \frac{1}{\sqrt{2}}|1\rangle|-\rangle \right)$$

$$(-1)^{f(0)} |+\rangle|-\rangle$$

global phase

sloped

$$-(-1)^{f(0)} = (-1)^{f(1)}$$

$$\frac{1}{\sqrt{2}} (-1)^{f(0)} |0\rangle|-\rangle - \frac{1}{\sqrt{2}} (-1)^{f(0)} |1\rangle|-\rangle$$

$$(-1)^{f(0)} \left(\frac{1}{\sqrt{2}}|0\rangle|-\rangle - \frac{1}{\sqrt{2}}|1\rangle|-\rangle \right)$$

$$(-1)^{f(0)} |-\rangle|-\rangle$$

global phase

U_f (Phase Kickback)

$$|x\rangle|-\rangle \rightarrow (-1)^{f(x)} |x\rangle|-\rangle$$

standard basis

- 2 cases
- $f(0) = f(1)$ (flat)
 - $f(0) \neq f(1)$ (sloped)

H ↓

$|0\rangle|-\rangle$

↓ Δ

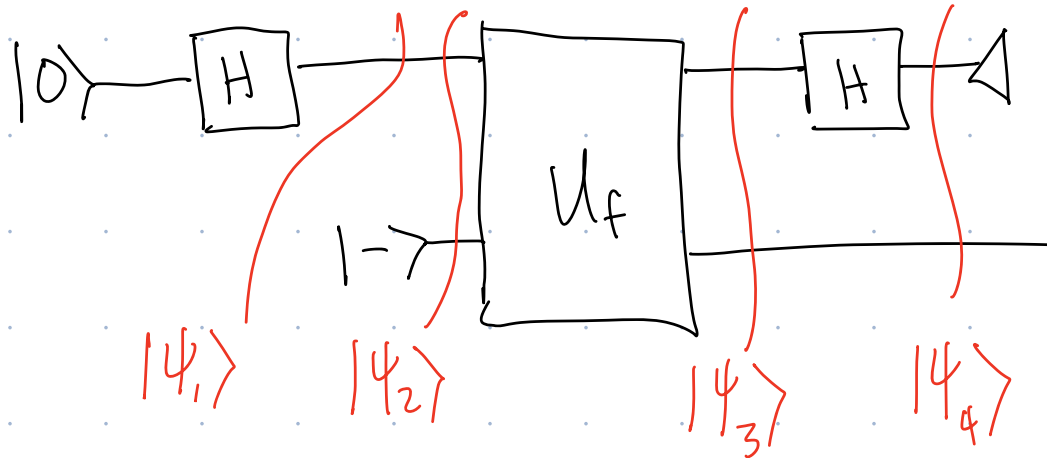
100% → outcome
 $|0\rangle$

↓

$|1\rangle|-\rangle$

100% → outcome
 $|1\rangle$

Deutsch's Alg:



Phase Kickback

