QUBITS!

Goals

- Describe qubits + quantum measurement using kets ✓
- Connect ket notation to physical intuition
- Analyze novel situations using kets

Announcements

- Quiz, rinse, repeat
- Posted notes

Exit Tickets

b'', d'' made public? X

classical computer measures photons?

how Eve gets info on s'?

how long L?

hash?

Eve's best strategy?

Lucky Eve?

why photons

single photon possible

used in practice?

Qubit = quantum bit

Single photon polarization ← e^- energy, e^- mag spin, superconducting can encode one qubit

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Some familiar qubit states

Photon	Ket Notation	English
↕	$\lvert 0\rangle$	ket 0, 0 state
↔	$\lvert 1\rangle$	ket 1, 1 state
⤢	$\lvert +\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle + \frac{1}{\sqrt{2}}\lvert 1\rangle$	plus state
⤡	$\lvert -\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle - \frac{1}{\sqrt{2}}\lvert 1\rangle$	minus state

$\{\lvert 0\rangle, \lvert 1\rangle\} \Rightarrow$ Standard basis

$\{\lvert +\rangle, \lvert -\rangle\} \Rightarrow$ Hadamard Basis

Why "Qubit"?

- Any state can be written with $\lvert 0\rangle, \lvert 1\rangle$
- Basic measurement has 2 outcomes

Qubit = quantum bit

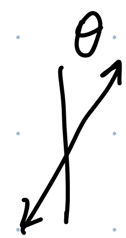
Single photon polarization can encode one qubit ← also e^- energy, e^- mag spin, ...

Some familiar qubit states

Photon	Ket Notation	English
$\updownarrow$	$\lvert 0\rangle$	ket 0, 0 state
$\leftrightarrow$	$\lvert 1\rangle$	ket 1, 1 state
$\nearrow\!\!\!\swarrow$	$\lvert +\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle + \frac{1}{\sqrt{2}}\lvert 1\rangle$	plus state
$\nwarrow\!\!\!\searrow$	$\lvert -\rangle = \frac{1}{\sqrt{2}}\lvert 0\rangle - \frac{1}{\sqrt{2}}\lvert 1\rangle$	minus state

$\{\lvert 0\rangle, \lvert 1\rangle\} \Rightarrow$ Standard basis states

$\{\lvert +\rangle, \lvert -\rangle\} \Rightarrow$ Hadamard basis states



How many qubit states are there?

A) 2 B) 4 C) Countably ∞ $\boxed{\text{D) Uncountably } \infty}$

Qubit State:

ket psi

state psi → $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$

$a_0, a_1 \in \mathbb{C}$

"amplitudes"

Normalization: $|a_0|^2 + |a_1|^2 = 1$

ex: $|0\rangle \Rightarrow a_0 = 1 \quad a_1 = 0$

$|\nearrow\rangle \Rightarrow a_0 = \frac{1}{\sqrt{2}} \quad a_1 = \frac{1}{\sqrt{2}}$

$\Rightarrow a_0 = \cos 30° \quad a_1 = \sin 30°$

Often convenient to think as a vector in $\mathbb{C}^2$

$$|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

"Superposition" - state that is a combination of standard basis state

Qubit Measurement

Represented mathematically by orthonormal pair of kets

$= \{ |0\rangle, |1\rangle \}$

$M = \{ |\phi_1\rangle, |\phi_2\rangle \}$

Measurement (basis)

outcomes

ortho: $\langle \phi_1 | \phi_2 \rangle = 0$

normal: $\langle \phi_1 | \phi_1 \rangle = 1$

$\langle \phi_2 | \phi_2 \rangle = 1$

If measure state $|\psi\rangle$ with $M = \{ |\phi_1\rangle, |\phi_2\rangle \}$:

- With probability $|\langle \phi_1 | \psi \rangle|^2$ get outcome $|\phi_1\rangle$, $|\psi\rangle$ collapses to $|\phi_1\rangle$
- With probability $|\langle \phi_2 | \psi \rangle|^2$ get outcome $|\phi_2\rangle$, $|\psi\rangle$ collapses to $|\phi_2\rangle$

Can't control outcome of measurement — Random!

Bras

If $|\psi\rangle = a_0|0\rangle + a_1|1\rangle \overset{\text{Bra}}{\Longrightarrow} \langle\psi| = a_0^*\langle 0| + a_1^*\langle 1|$

$$\boxed{\begin{array}{l}\text{Conj. transpose} \\ \langle\psi| = (a_0^*,\ a_1^*)\end{array}}$$

$$\frac{i}{\sqrt{2}}|\phi_1\rangle - \frac{i}{\sqrt{2}}|\phi_2\rangle \overset{\text{Bra}}{\Longrightarrow} \frac{-i}{\sqrt{2}}\langle\phi_1| + \frac{i}{\sqrt{2}}\langle\phi_2|$$

Brakets / Inner Products

$$\langle\cdot|\cdot\rangle = (\cdots)\begin{pmatrix}\vdots\\\vdots\end{pmatrix} = \text{number}$$

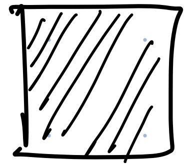
Basic Rules:

- $\langle 0|0\rangle = 1 \quad \langle 1|1\rangle = 1 \quad \langle 0|1\rangle = 0 \quad \langle 1|0\rangle = 0$
- Numbers commute bras/kets: $\left(\frac{1}{\sqrt{2}}\langle 0|\right)\left(\frac{1}{\sqrt{3}}|0\rangle\right) = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}\cdot\langle 0|0\rangle = \frac{1}{\sqrt{6}}$

  Numbers = scalars
- Braket Distributes
- If $\langle\psi|\phi\rangle = 0 \Longleftrightarrow \langle\phi|\psi\rangle = 0 \Rightarrow |\psi\rangle, |\phi\rangle$ are orthogonal states

Example

$\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

$\circlearrowright \rightsquigarrow$ 

$M = \{|+\rangle, |-\rangle\} \rightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Probability of no photon exiting?

$$|\langle - | \circlearrowright \rangle|^2 = \left| \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)\right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \overset{1}{\langle 0|0\rangle} + \frac{1}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} \overset{0}{\cancel{\langle 0|1\rangle}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \overset{0}{\cancel{\langle 1|0\rangle}} - \frac{1}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} \overset{1}{\langle 1|1\rangle} \right|^2$$

$$= \left| \frac{1}{2} - \frac{i}{2} \right|^2$$

$$= \left(\frac{1}{2} - \frac{i}{2}\right) \cdot \left(\frac{1}{2} + \frac{i}{2}\right) = \frac{1}{4} - \frac{i}{4} + \frac{i}{4} + \frac{1}{4} = \frac{1}{2}$$

1:

$\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$

Right circ. , Left circ.

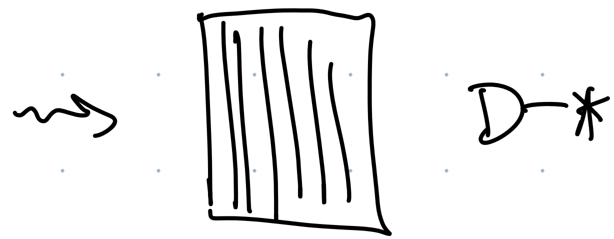
$M = \left\{ \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle \right\}$

Right circularly polarized filter.

What is probability a photon emerges + what polarization?

2.

$i\cos 30°|0\rangle + \sin 30°|1\rangle$

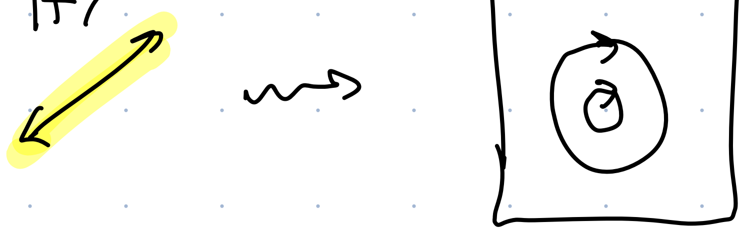


What is prob of detection / no detection?

3. If state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$, what is $\langle\psi|\psi\rangle$?

1:

$|+\rangle$



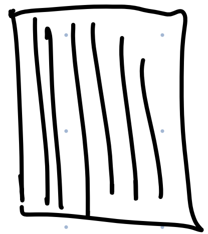
Right circ. , Left circ.

$$M = \left\{ \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{i}{\sqrt{2}}|1\rangle,\ \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{i}{\sqrt{2}}|1\rangle \right\}$$

Right circularly polarized filter.

What is probability a photon emerges + what polarization?

2.

$i\cos 30°|0\rangle + \sin 30°|1\rangle \rightsquigarrow$  

What is prob of detection / no detection?

Detection:

No detection

3. If state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$, what is $\langle\psi|\psi\rangle$?

"Superposition" - combination of standard basis states

ex: $|+\rangle$ is a superposition of 0 and 1

Bras

If $|\psi\rangle = a_0|0\rangle + a_1|1\rangle \Rightarrow$

Brakets/Inner Products

Basic Rules:

$\langle 0|0\rangle = 1 \qquad \langle 1|1\rangle = 1 \qquad \langle 0|1\rangle = 0 \qquad \langle 1|0\rangle = 1$

ex:

Inner product $\langle \psi|+\rangle$ if $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$:

comp. conj

$$\langle \psi|+\rangle = \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{2}\langle 0|0\rangle + \frac{1}{2}\langle 0|1\rangle + \frac{i}{2}\langle 1|0\rangle + \frac{i}{2}\langle 1|1\rangle \quad \text{Distribute}$$

$$= \frac{1}{2} + 0 + 0 + \frac{i}{2} \quad \text{Rules}$$

$$= \frac{1+i}{2}$$

If $\langle \psi|\phi\rangle = 0 \Leftrightarrow |\psi\rangle, |\phi\rangle$ are "orthogonal"

Q: If $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$, what is $\langle\psi|\psi\rangle$?

$$\langle\psi|\psi\rangle = \left(a_0^*\langle 0| + a_1^*\langle 1|\right)\left(a_0|0\rangle + a_1|1\rangle\right)$$

$$= a_0 a_0^* \langle 0|0\rangle + a_0^* a_1 \langle 0|1\rangle + a_1^* a_0 \langle 1|0\rangle + a_1^* a_1 \langle 1|1\rangle$$

$$= a_0 a_0^* + a_1 a_1^*$$

$$= |a_0|^2 + |a_1|^2$$

$$= 1$$

$|a_0|^2 = a_0 a_0^*$

A properly normalized state has inner product 1 with itself.

Qubit Measurement

Represented mathematically by an $\underbrace{\text{ortho}}_{\downarrow}\underbrace{\text{normal}}$ pair of kets

$M = \{|\phi_1\rangle, |\phi_2\rangle\}$ $\qquad \langle\phi_1|\phi_2\rangle = 0 \qquad \langle\phi_1|\phi_1\rangle = 1$

Measurement $\qquad$ meas. outcomes $\qquad \langle\phi_2|\phi_1\rangle = 0 \qquad \langle\phi_2|\phi_2\rangle = 1$

according to laws of q. mechanics!

If measure state $|\psi\rangle$ with $M = \{|\phi_1\rangle, |\phi_2\rangle\}$:

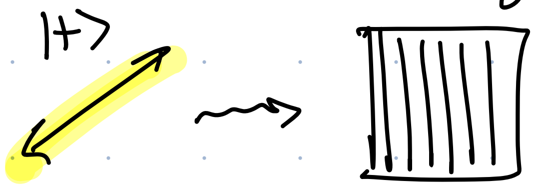
(asking: is $|\psi\rangle$ in state $|\phi_1\rangle$ or $|\phi_2\rangle$, ex asks $\updownarrow$ or $\leftrightarrow$)

- With probability $|\langle\phi_1|\psi\rangle|^2$ get outcome $|\phi_1\rangle$,
  $|\psi\rangle$ collapses to $|\phi_1\rangle$

- With probability $|\langle\phi_2|\psi\rangle|^2$ get outcome $|\phi_2\rangle$,
  $|\psi\rangle$ collapses to $|\phi_2\rangle$

Can't control outcome of measurement — Random!

Example



$M = \{|0\rangle, |1\rangle\}$

$|0\rangle$ outcome $\rightarrow \updownarrow$

$|1\rangle$ outcome $\rightarrow \leftrightarrow$

Probability of no photon exiting (outcome $|1\rangle$)

$$|\langle 1|+\rangle|^2 = \left|\langle 1|\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\right|^2$$

$$= \left|\tfrac{1}{\sqrt{2}}\langle 1|0\rangle + \tfrac{1}{\sqrt{2}}\langle 1|1\rangle\right|^2$$

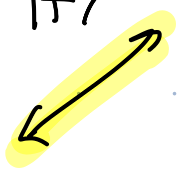
$$= \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

$\Rightarrow$ $|+\rangle$ collapses to $|1\rangle = \leftrightarrow$, which gets blocked by filter and so no photon comes out!

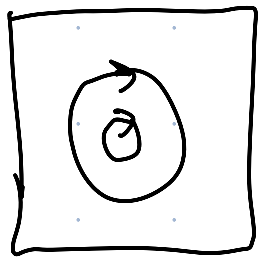
In each case:

- What is the probability a photon exits?
- What is the polarization of the photon if it exits?

• $|+\rangle$



⇝



Right circ. , Left circ.

$$M = \left\{ \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle,\ \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle \right\}$$

Right circularly polarized filter. Used for 3D movie glasses

$$\left| \frac{1}{\sqrt{2}}\left(\langle 0| - i\langle 1|\right) \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \right|^2$$

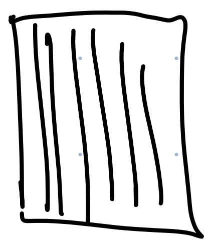
$$\left| \frac{1}{2}(1-i) \right|^2 = \frac{1}{2}(1-i)\cdot\frac{1}{2}(1+i)$$

$$= \frac{1}{4}(1+1) = \frac{1}{2}$$

•

$i\cos 30^\circ|0\rangle + \sin 30^\circ|1\rangle$

⇝



$$M = \{|0\rangle, |1\rangle\}$$

$$\left|\langle 0|\left(i\cos 30^\circ|0\rangle + \sin 30^\circ|1\rangle\right)\right|^2$$

$$= |\cos 30^\circ|^2$$

$$= 0.87$$

Standard basis → amplitudes squared

Prob no exit: $$\left|\langle 1|\left(i\cos 30^\circ|0\rangle + \sin 30^\circ|1\rangle\right)\right|^2$$

$$= |i\cos 30^\circ\langle 1|0\rangle + \sin 30^\circ\langle 1|1\rangle|^2$$

$$= |\sin 30^\circ|^2$$