Grover's Search Algorithm

Learning Goals

2nd most famous

- Practice analyzing a new q. algorithm w/ geometric technique
- Learn an important q. algorithmic subroutine

Exit Tickets

- Period Finding → OH
- Upcoming exam:
  - If you did not attempt no need to come to OH
  - No QC4
  - You are responsible for knowing which targets to solve

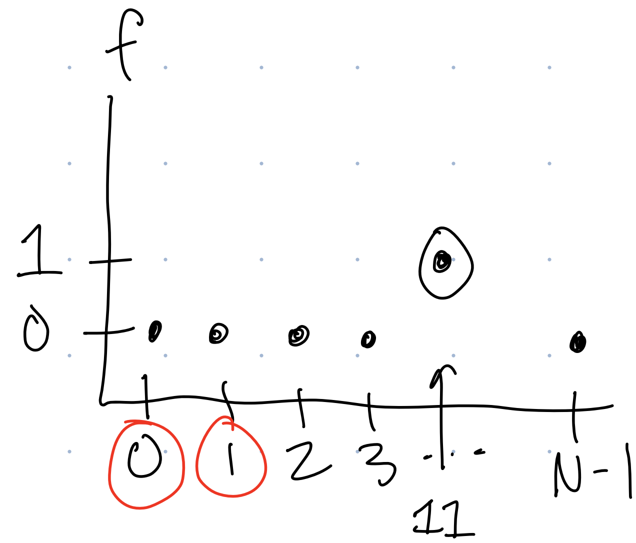
    Grade → Learning Mastery (test student)
- Thanks again for the conversation

Search Problem

Input: $f:\{0,1,2,\dots,N-1\}\to\{0,1\}$ s.t.

- $\exists\ s: f(s)=1$
- for all other $x\neq s$, $f(s)=0$

Output: s ← "marked item"



What is the classical query complexity (deterministic)? $N-1$

A) $O(1)$ B) $O(\log N)$ C) $O(N)$ D) $O(2^N)$

What is the classical query complexity (probabilistic)? $\frac{2N}{3}$

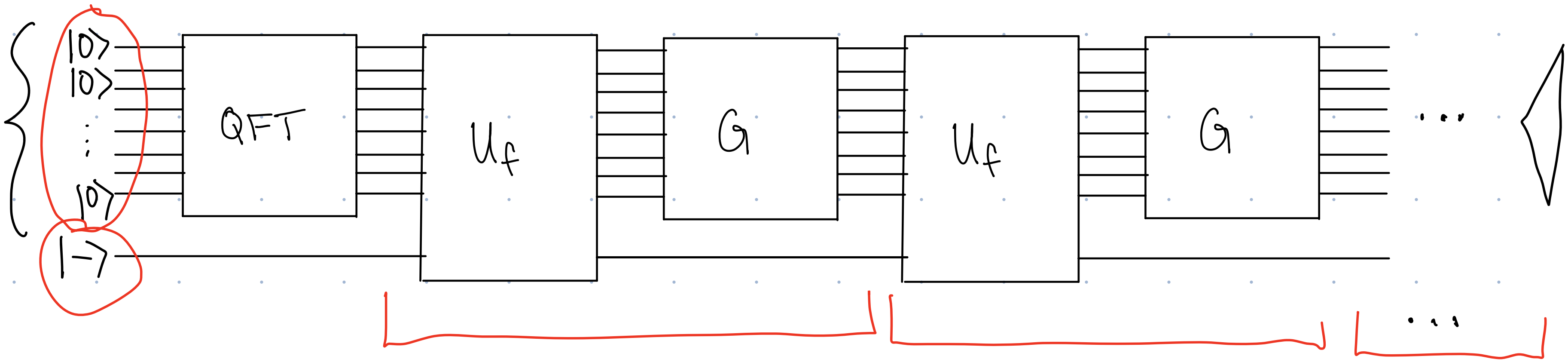
A) $O(1)$ B) $O(\log N)$ C) $O(N)$ D) $O(2^N)$

Grover's Algorithm (Quantum Search Alg)

N-dim state

$\log_2 N$

1



How many qubits are needed to search N possible inputs to f?

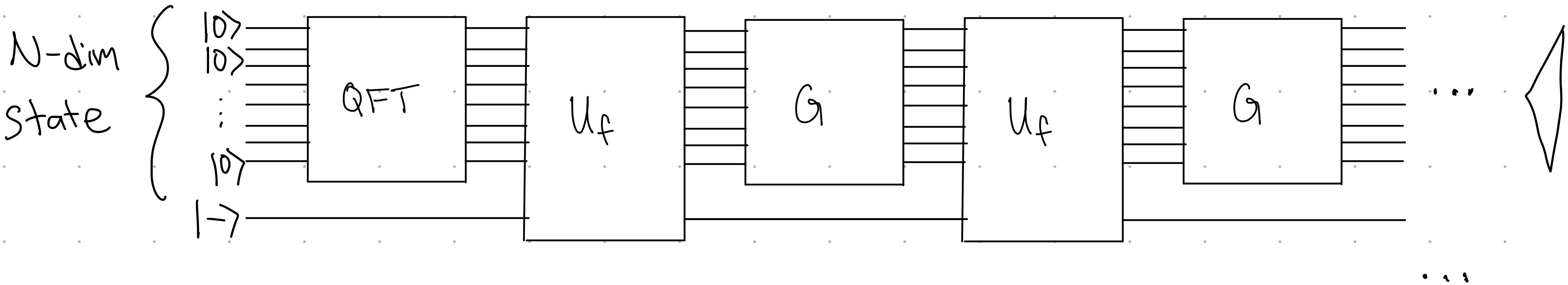
A) $\log_2 N$

B) $\log_2 N + 1$

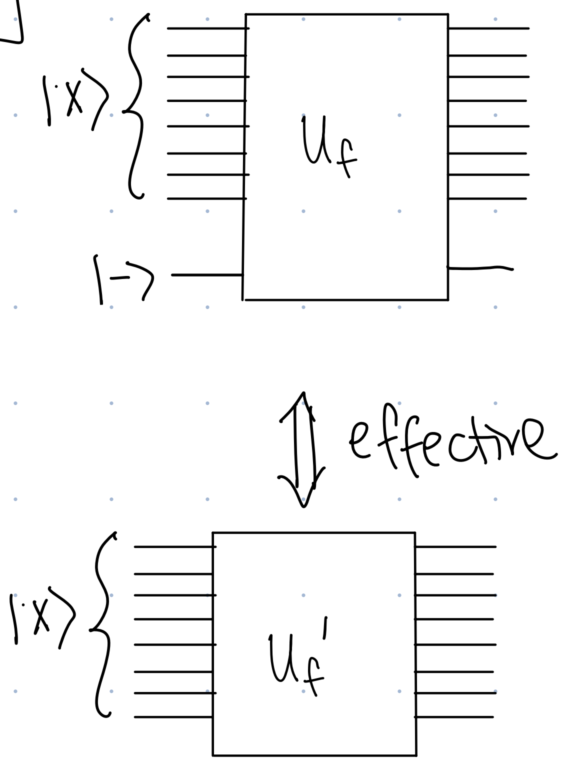
C) N

D) N + 1

Grover's Algorithm (Quantum Search Alg)



U_f:



Phase Kickback: (If $|x\rangle$ is standard basis.)

$$U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$$

$$U_f|x\rangle|-\rangle = \begin{cases} -|x\rangle|-\rangle & \text{if } x = s \\ |x\rangle|-\rangle & \text{else } x \neq s \end{cases}$$

"outer product"

$$U_f' = I - 2|s\rangle\langle s|$$

Standard basis

$$U_f'|s\rangle = (I - 2|s\rangle\langle s|)|s\rangle$$
$$= I|s\rangle - 2|s\rangle\overbrace{\langle s|s\rangle}^{1}$$
$$= |s\rangle - 2|s\rangle = -|s\rangle$$

$$U_f'|x \neq s\rangle = (I - 2|s\rangle\langle s|)|x \neq s\rangle$$
$$= I|x \neq s\rangle - \cancel{2|s\rangle\overbrace{\langle s|x \neq s\rangle}^{0}}$$
$$= |x \neq s\rangle$$

$\boxed{G}$ (Grover Diffusion Operator)

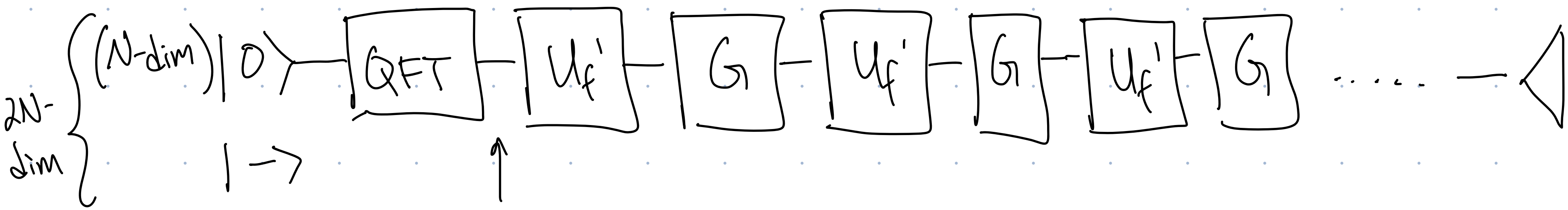
"alpha"

$$G = -I + 2|\alpha\rangle\langle\alpha| \quad \text{where} \quad |\alpha\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle$$

$$G|\gamma\rangle = \begin{cases} |\gamma\rangle & \text{if } |\gamma\rangle = |\alpha\rangle \\ -|\gamma\rangle & \text{if } \langle\gamma|\alpha\rangle = 0 \end{cases}$$

"gamma"

Grovers Alg: (Effective)



While the state is 2N-Dimensional, througout the alg, state only stays in 2 of those 2N dimensions.

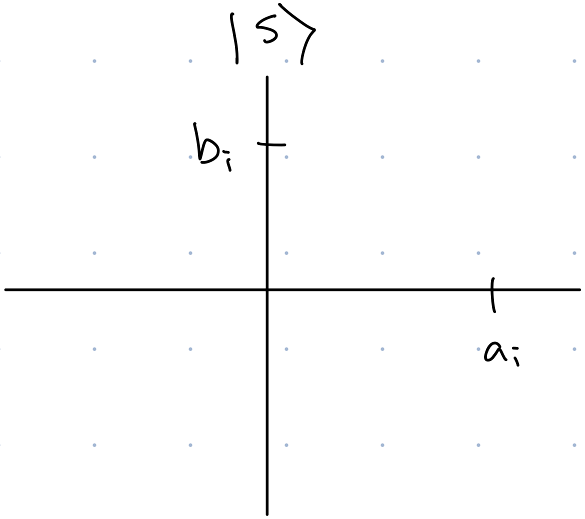
In other words, we can describe the state as a superposition of 2 states instead of N.

In particular:

"beta"

$$|\psi_i\rangle = a_i|\beta\rangle + b_i|s\rangle \quad \text{where} \quad |\beta\rangle = \frac{1}{\sqrt{N-1}}\sum_{i\neq s}|i\rangle$$

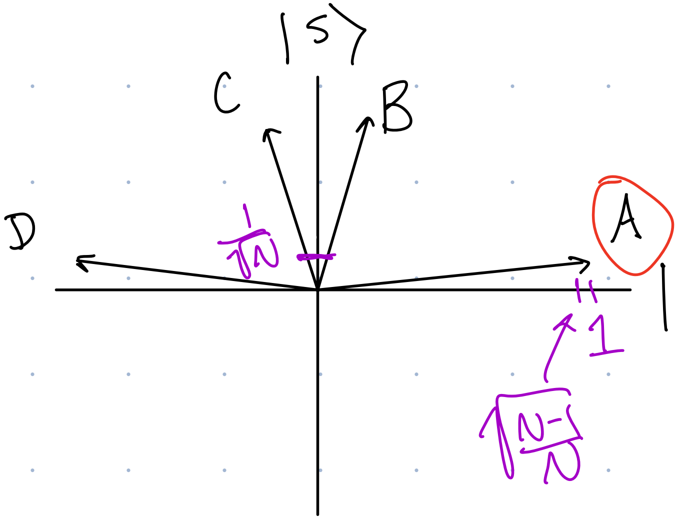
We can express $|\psi_i\rangle = a_i|\beta\rangle + b_i|s\rangle$ as a vector in 2D:



$$|\beta\rangle = \frac{1}{\sqrt{N-1}} \sum_{i\neq s} |i\rangle$$

$\boxed{\text{QC3}}$

Where is $|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$? (assume N is big)



$$|\beta\rangle = \frac{1}{\sqrt{N-1}} \sum_{i\neq s} |i\rangle$$

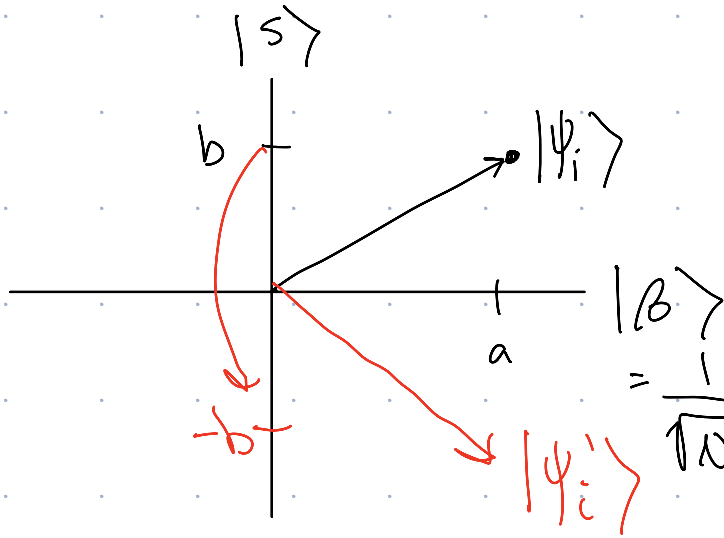
$$|\alpha\rangle = a\left(\frac{1}{\sqrt{N-1}} \sum_{i\neq s} |i\rangle\right) + b|s\rangle$$

≈ 1

$$\boxed{\frac{\cancel{\sqrt{N-1}}}{\sqrt{N}}}\left(\frac{1}{\cancel{\sqrt{N-1}}} \sum_{i\neq s} |i\rangle\right) + \boxed{\frac{1}{\sqrt{N}}}|s\rangle$$

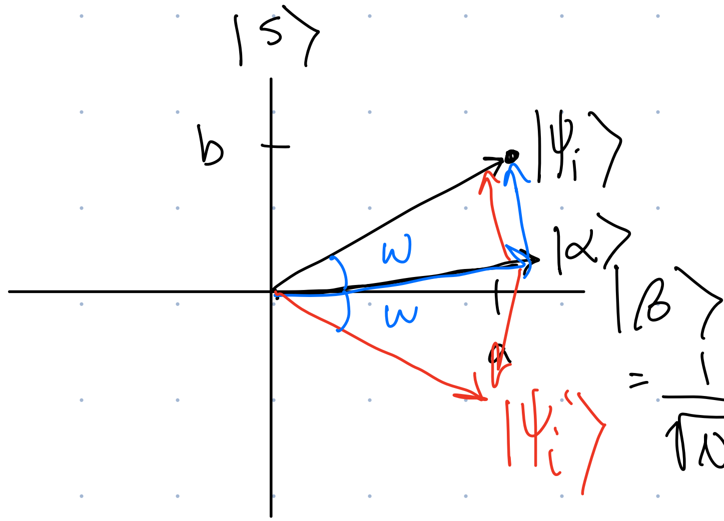
$$|\alpha\rangle \doteq \frac{1}{\sqrt{N}} \sum_{i\neq s} |i\rangle + \frac{1}{\sqrt{N}}|s\rangle$$

Effect of $U_f' = I - 2|s\rangle\langle s|$



$$= \frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle$$

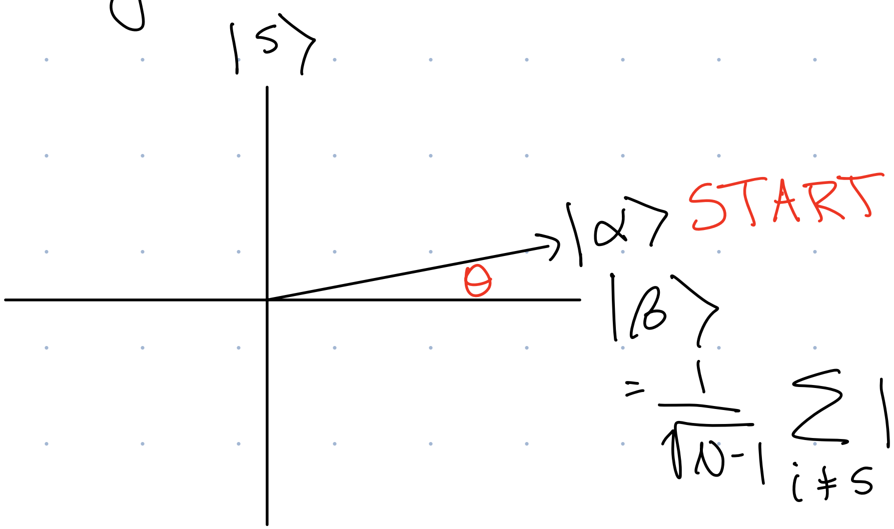
Effect of $G = -I + 2|\alpha\rangle\langle\alpha|$



$$= \frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle$$

How many iterations before state becomes $|s\rangle$?

$(U_f', G, U_f', G, \ldots)$

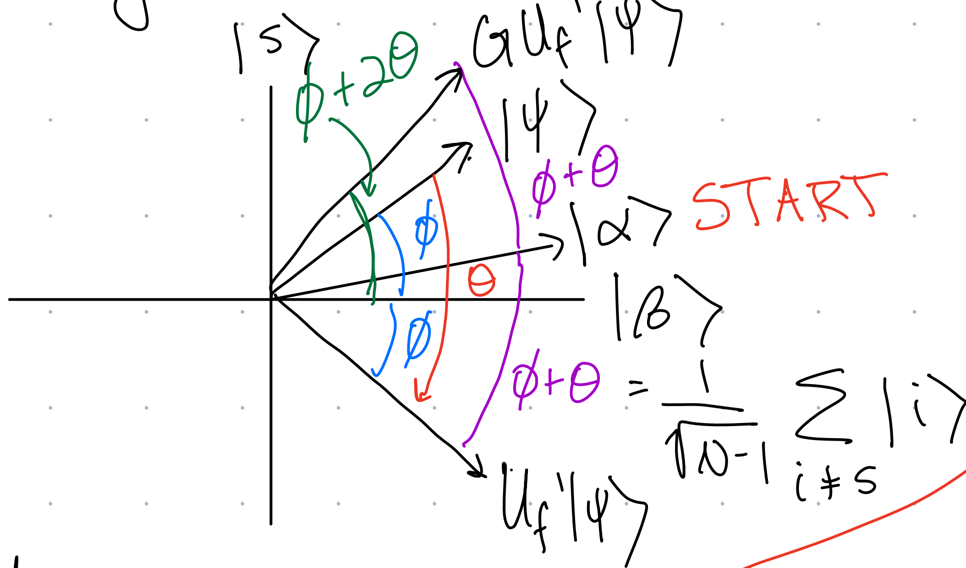


$$= \frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle$$

$\tan^{-1}(x) \approx x$

for $x \ll 1$

How many iterations before state becomes $|s\rangle$?



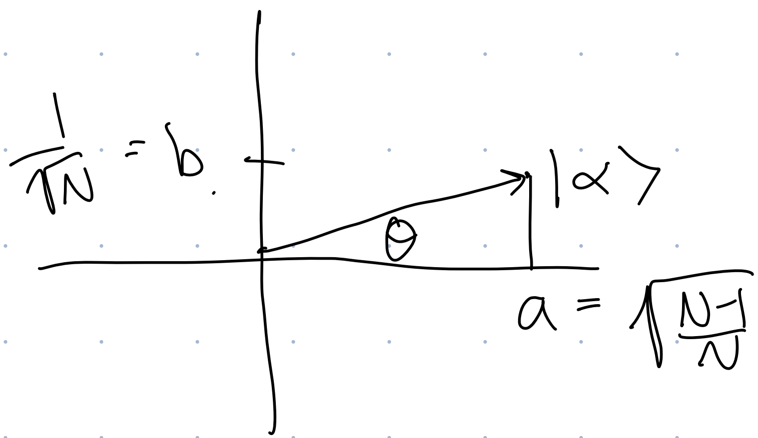
$$\boxed{\frac{\frac{\pi}{2} - \theta}{2\theta}}$$

total angular distance

each iteration

$$= \frac{\pi}{4\theta} - \frac{1}{2}$$

\# of iterations



$$\tan^{-1}(\tan(\theta)) = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\frac{1}{\sqrt{N}}}{\frac{\sqrt{N-1}}{\sqrt{N}}}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{N-1}}\right)$$

$$\theta \approx \frac{1}{\sqrt{N-1}} \approx \frac{1}{\sqrt{N}}$$

\# iterations:

$$\approx \frac{1}{4}\pi\sqrt{N} - \frac{1}{2}$$

$$O(\sqrt{N})$$

Note

Searches for input to a function — not a search through data.

U_f ?

Application

Suppose have classical alg that succeeds with prob p.

$$f(c) = \begin{cases} 1 & \text{if coin flips } c \text{ lead to success} \\ 0 & \text{else} \end{cases}$$

Can create a quantum alg that searches for set of random choices that causes alg to succeed. $O\left(\frac{1}{\sqrt{p}}\right)$

ex: Best classical 3-SAT alg: $p = \left(\frac{3}{4}\right)^n \rightarrow O\left(\frac{1}{\sqrt{p}}\right) \rightarrow O\left(\left(\frac{4}{3}\right)^{n/2}\right)$

$\xrightarrow{\text{Grover}}$ Quant. Alg with runtime:

$$O\left(\left(\frac{2}{\sqrt{3}}\right)^n\right)$$

Can do for any prob. alg. $\rightarrow$ not that exciting theoretically
Not clear how to do for data