

Grover's Search Algorithm

Learning Goals

- Practice analyzing a new q. algorithm w/ geometric technique
- Learn an important q. algorithmic subroutine

2nd most famous
↓

Exit Tickets

- Period Finding $\rightarrow$ OH
 - Upcoming exam:
 - If you did not attempt no need to come to OH
 - No Qc4
 - You are responsible for knowing which targets to solve
- Grade $\rightarrow$ Learning Mastery (test student)
- Thanks again for the conversation

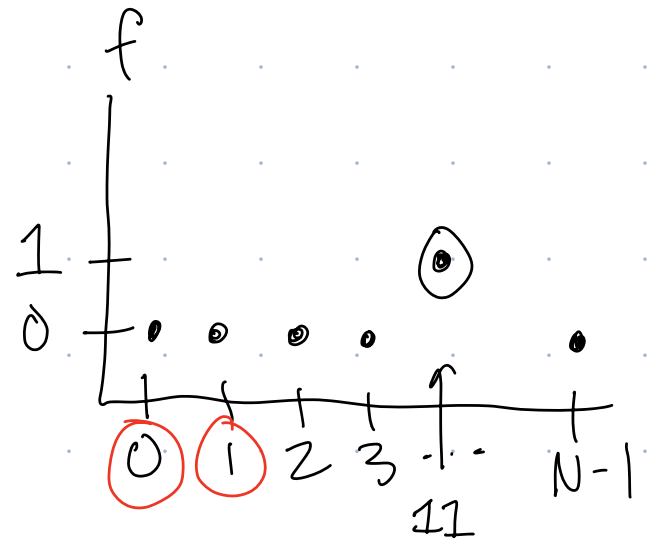
Search Problem

Input: $f: \{0, 1, 2, \dots, N-1\} \rightarrow \{0, 1\}$ s.t.

• $\exists s: f(s)=1$

• for all other $x \neq s$, $f(x)=0$

Output: $s \leftarrow$ "marked item"



What is the classical query complexity (deterministic)? $N-1$

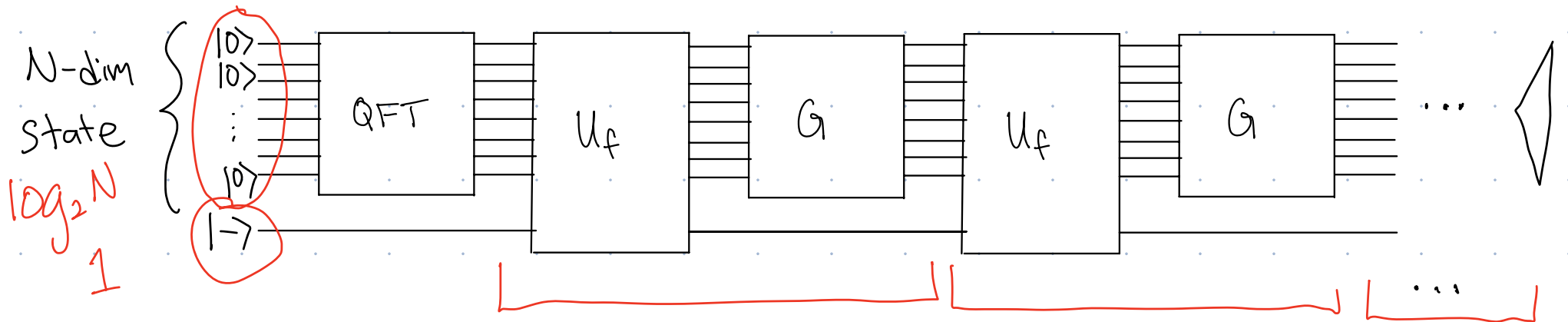
A) $O(1)$ B) $O(\log N)$ C) $O(N)$ D) $O(2^N)$

What is the classical query complexity (probabilistic)? $\frac{2N}{3}$

A) $O(1)$ B) $O(\log N)$ C) $O(N)$ D) $O(2^N)$

Grover's Algorithm

(Quantum Search Alg)



How many qubits are needed to search N possible inputs to f ?

A) $\log_2 N$

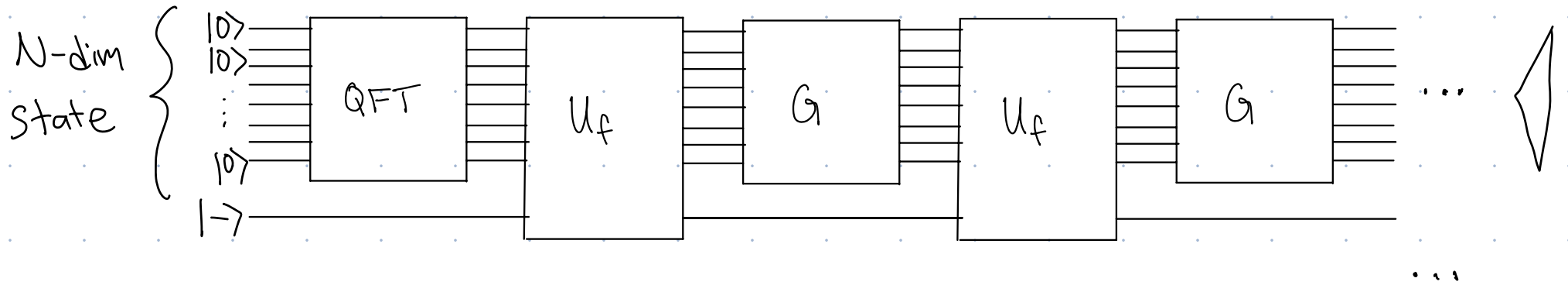
B) $\log_2 N + 1$

C) N

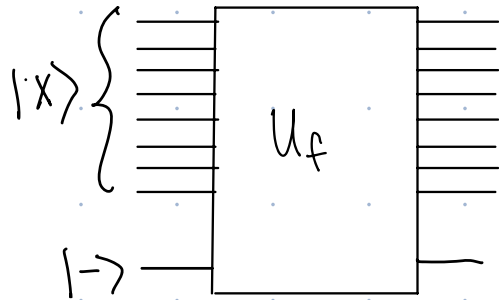
D) $N + 1$

Grover's Algorithm

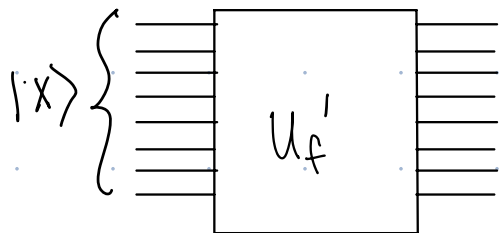
(Quantum Search Alg)



U_f :



↕ effective



Phase Kickback: (If $|x\rangle$ is standard basis.)

$$U_f |x\rangle |-\rangle = (-1)^{f(x)} |x\rangle |-\rangle$$

$$U_f |x\rangle |-\rangle = \begin{cases} -|x\rangle |-\rangle & \text{if } x = s \\ |x\rangle |-\rangle & \text{else } x \neq s \end{cases}$$

"outer product"

$$U_f' = I - 2|s\rangle\langle s|$$

$$\begin{aligned}
 U_f'|s\rangle &= (I - 2|s\rangle\langle s|)|s\rangle \\
 &= I|s\rangle - 2|s\rangle\langle s|s\rangle \\
 &= |s\rangle - 2|s\rangle = -|s\rangle
 \end{aligned}$$

Standard basis

Annotations: A red arrow points from the '1' in the second term to the '1' in the first term. A blue arrow points from the 'Standard basis' text to the '1' in the first term.

$$\begin{aligned}
 U_f'|x \neq s\rangle &= (I - 2|s\rangle\langle s|)|x \neq s\rangle \\
 &= I|x \neq s\rangle - \cancel{2|s\rangle\langle s|x \neq s\rangle} \\
 &= |x \neq s\rangle
 \end{aligned}$$

Annotations: A red arrow points from the '0' in the second term to the '0' in the first term. A blue arrow points from the 'Standard basis' text to the '0' in the first term.

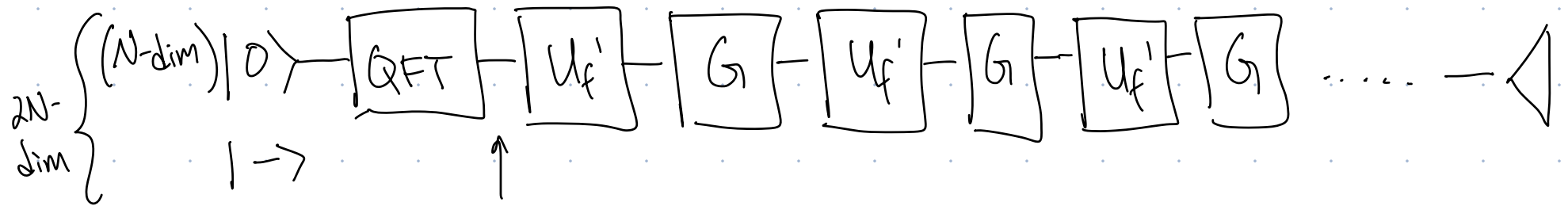
G (Grover Diffusion Operator) "alpha"

$$G = -I + 2|\alpha\rangle\langle\alpha| \quad \text{where} \quad |\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle$$

$$G|\gamma\rangle = \begin{cases} |\gamma\rangle & \text{if } |\gamma\rangle = |\alpha\rangle \\ -|\gamma\rangle & \text{if } \langle\gamma|\alpha\rangle = 0 \end{cases}$$

"gamma"

Grovers Alg: (Effective)



While the state is $2N$ -Dimensional, throughout the alg, state only stays in 2 of those $2N$ dimensions.

In other words, we can describe the state as a superposition of 2 states instead of N .

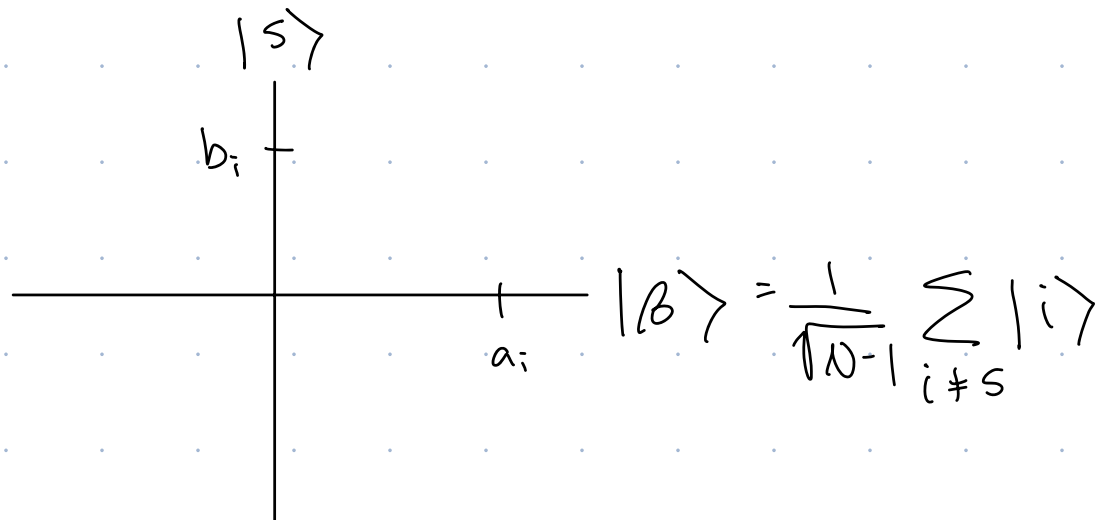
In particular:

$$|\psi_i\rangle = a_i |\beta\rangle + b_i |s\rangle$$

↙ "beta"

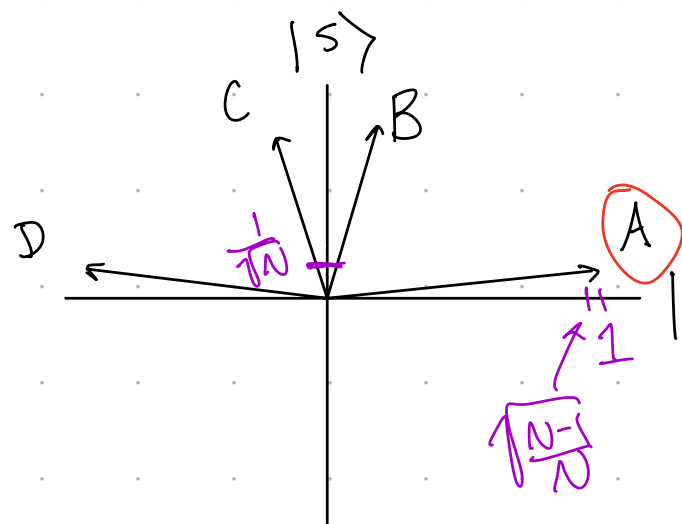
where $|\beta\rangle = \frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle$

We can express $|\psi_i\rangle = a_i|\beta\rangle + b_i|s\rangle$
as a vector in 2D:



Qc3

Where is $|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$? (assume N is big)



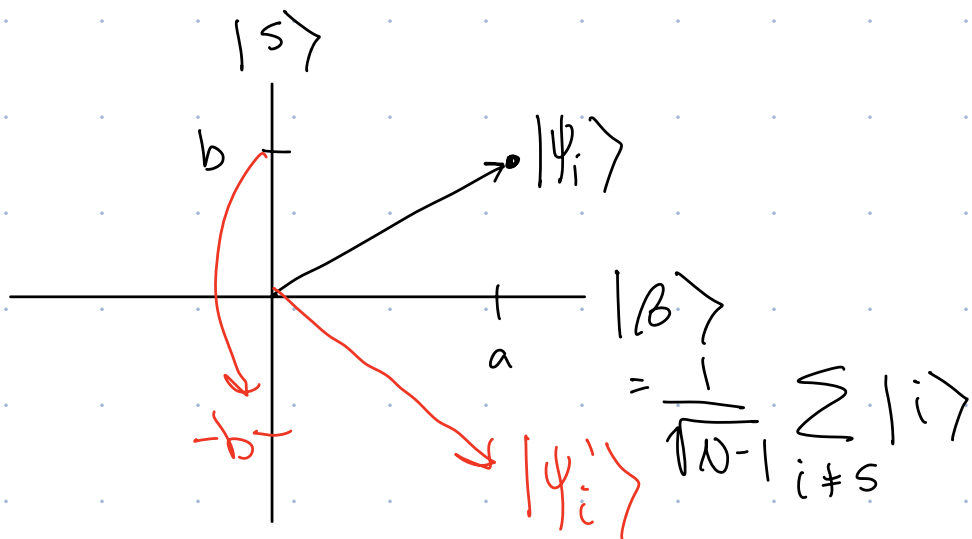
$$|\beta\rangle = \frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle$$

$$|\alpha\rangle = a \left(\frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle \right) + b |s\rangle$$

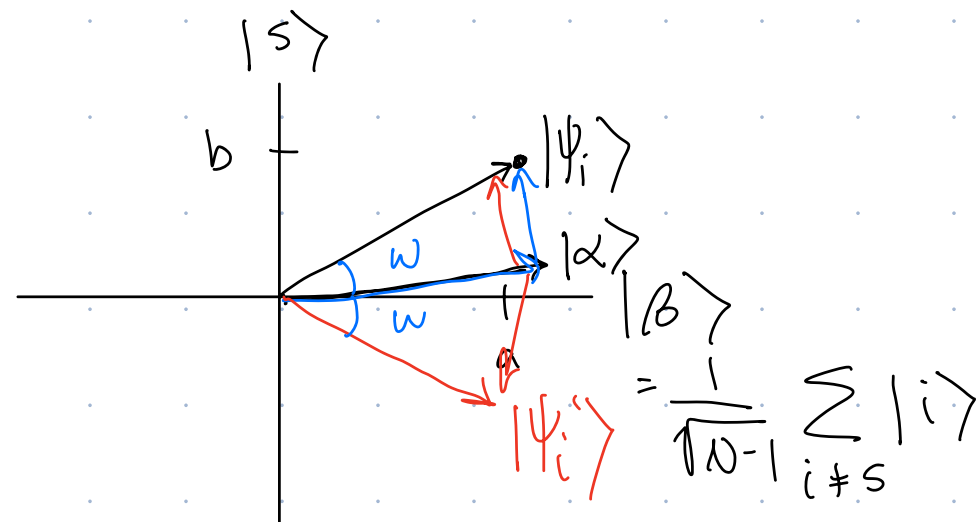
$$\approx \frac{1}{\sqrt{N}} \left(\frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle \right) + \frac{1}{\sqrt{N}} |s\rangle$$

$$|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{i \neq s} |i\rangle + \frac{1}{\sqrt{N}} |s\rangle$$

Effect of $U_f' = I - 2|s\rangle\langle s|$

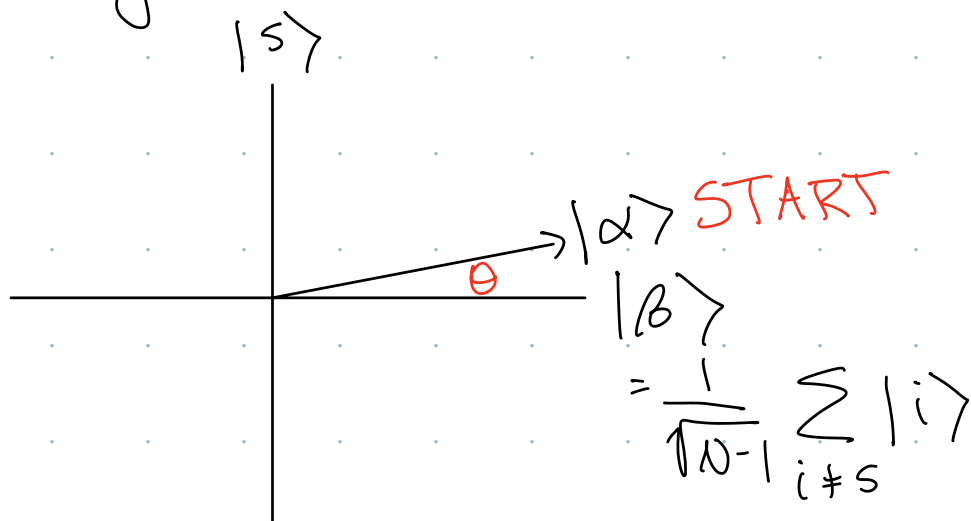


Effect of $G = -I + 2|\alpha\rangle\langle\alpha|$



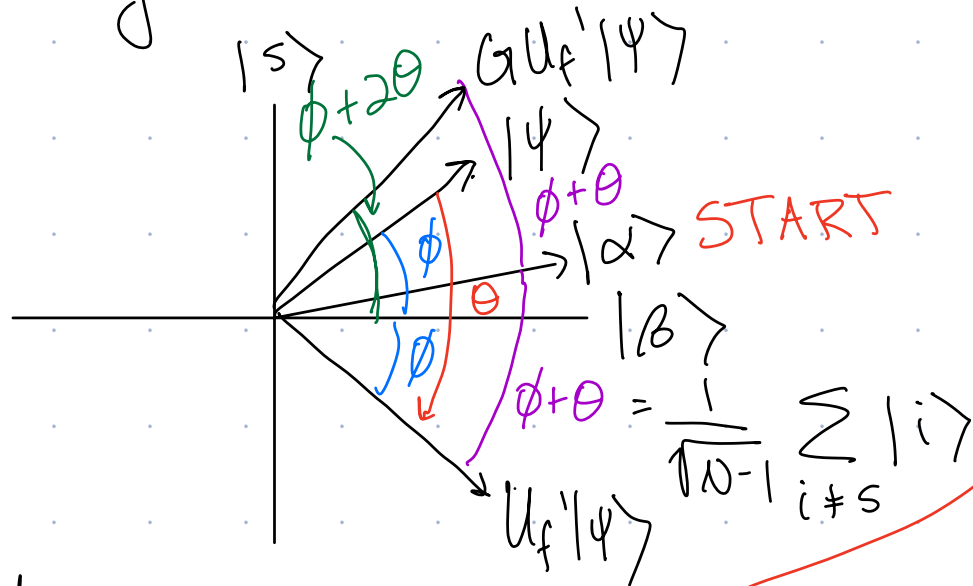
How many iterations before state becomes $|s\rangle$?

$(U_f', G, U_f', G, \dots)$



$$\tan^{-1}(x) \approx x \text{ for } x \ll 1$$

How many iterations before state becomes $|s\rangle$?

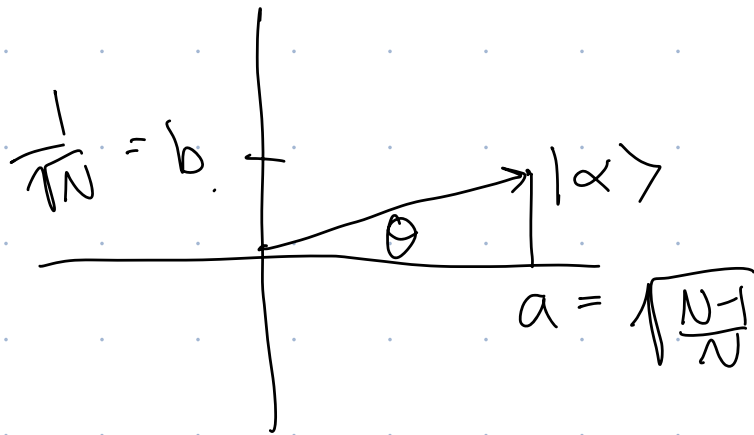


$$\boxed{\frac{\frac{\pi}{2} - \theta}{2\theta}}$$

total angular distance
each iteration

$\uparrow$
of iterations

$$= \frac{\pi}{4\theta} - \frac{1}{2}$$



$$\tan^{-1}(\tan(\theta)) = \frac{b}{a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\frac{1}{\sqrt{N}}}{\frac{\sqrt{N-1}}{\sqrt{N}}}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{N-1}}\right)$$

$$\theta \approx \frac{1}{\sqrt{N-1}} \approx \frac{1}{\sqrt{N}}$$

iterations:

$$\approx \frac{1}{4} \pi \sqrt{N} - \frac{1}{2}$$

$$O(\sqrt{N})$$

Note

Searches for input to a function — not a search through data.

Application

Suppose have classical alg that succeeds with prob p .

$$f(c) = \begin{cases} 1 & \text{if coin flips } c \text{ lead to success} \\ 0 & \text{else} \end{cases}$$

Can create a quantum alg that searches for set of random choices that causes alg to succeed.

ex: Best classical 3-SAT alg:

↳
Grover

Quant. Alg with runtime:

Can do for any prob. alg. $\rightarrow$ not that exciting theoretically
Not clear how to do for data