Grover's Search Algorithm

Learning Goals

2nd most famous

- Practice analyzing a new q. algorithm w/ geometric technique
- Learn an important q. algorithmic subroutine

Search Problem

Input: $f: \{0, 1, 2, \ldots, N-1\} \to \{0,1\}$

- $\exists\ s: f(s) = 1$
- $\forall\ x \neq s,\ f(s) = 0$

Output: s ← "marked item"

$f(10) = 1$

$f(0), f(1), f(2) \ldots f(100) = 0$

What is the classical query complexity (deterministic)?

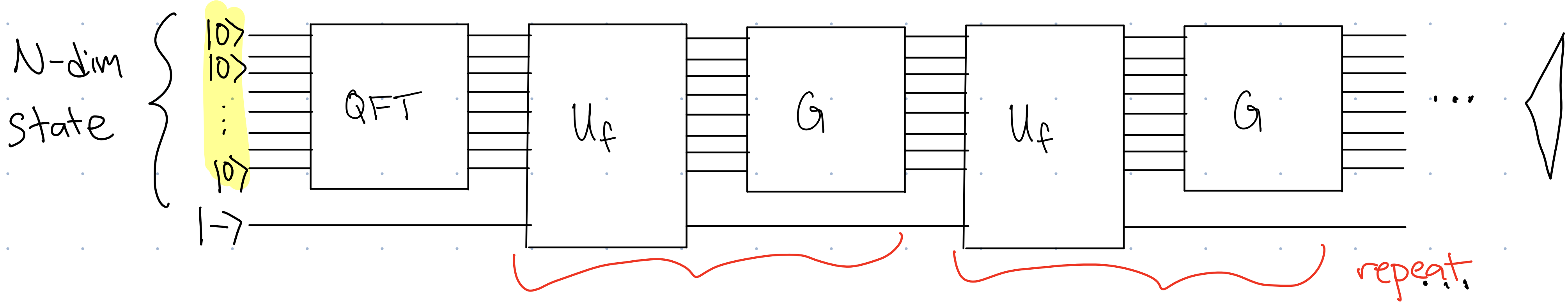
A) $O(1)$ B) $O(\log N)$ $\boxed{\text{C) } \Theta(N)}$ D) $O(2^N)$

What is the classical query complexity (probabilistic)?

A) $O(1)$ B) $O(\log N)$ $\boxed{\text{C) } \Theta(N)}$ D) $O(2^N)$

$\sim \frac{N}{2}$

Grover's Algorithm (Quantum Search Alg)



How many qubits are needed to search N possible inputs to f?

A) $\log_2 N$

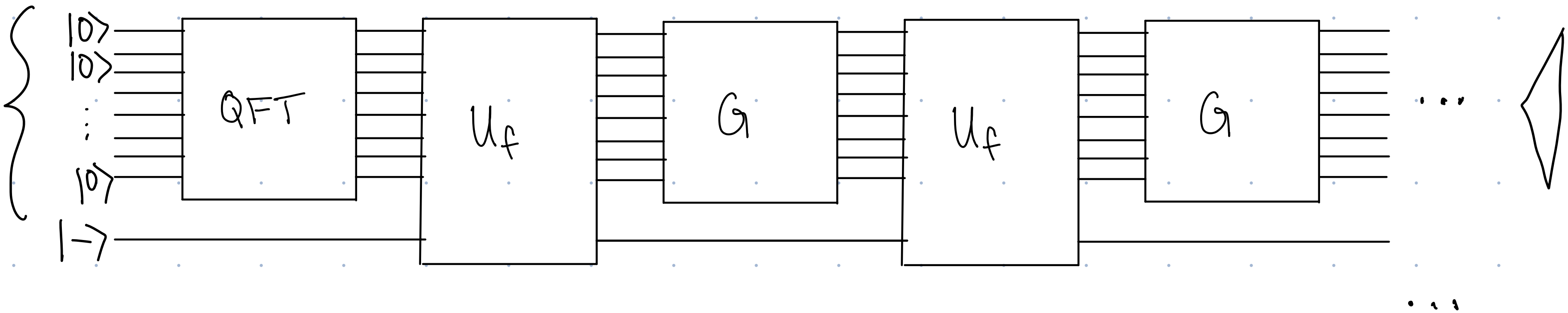
B) $\boxed{\log_2 N + 1}$

C) N

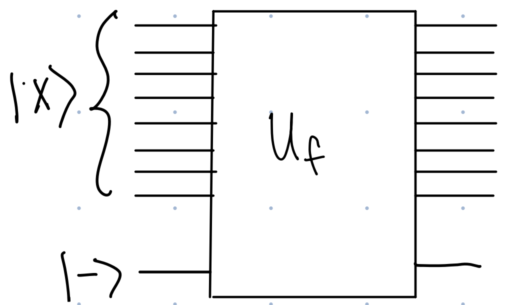
D) $N + 1$

Grover's Algorithm (Quantum Search Alg)

N-dim state

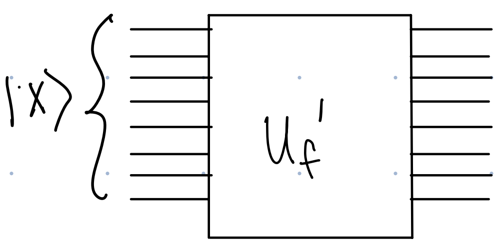


U_f:



effective

log₂N qubits



Phase Kickback

standard basis state x

$$U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$$

$$U_f|x\rangle|-\rangle = \begin{cases} -|x\rangle|-\rangle & \text{if } x = s \\ |x\rangle|-\rangle & \text{if } x \neq s \end{cases}$$

$$U_f' = I - 2|s\rangle\langle s|$$

$$U_f'|s\rangle = (I - 2|s\rangle\langle s|)|s\rangle$$
$$= I|s\rangle - 2|s\rangle\underbrace{\langle s|s\rangle}_{1}$$
$$= |s\rangle - 2|s\rangle = -|s\rangle$$

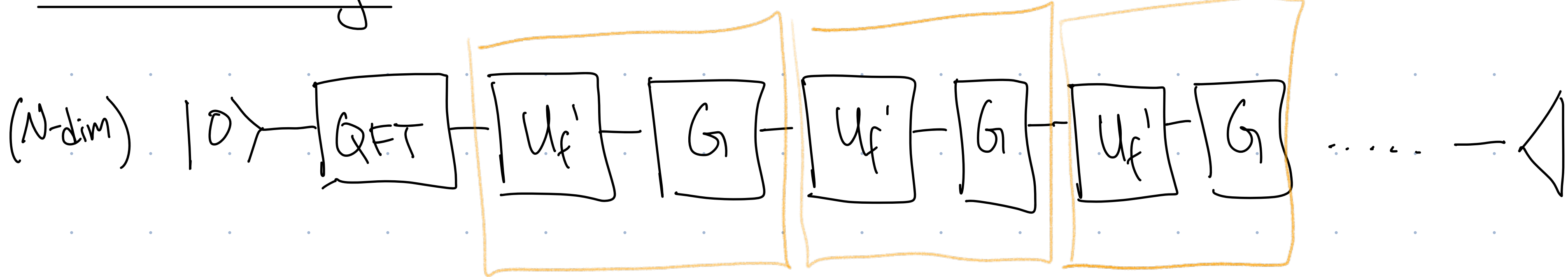
$$U_f'|x\neq s\rangle = (I - 2|s\rangle\langle s|)|x\neq s\rangle$$
$$= I|x\neq s\rangle - 2|s\rangle\underbrace{\langle s|x\neq s\rangle}_{0}$$
$$= I|x\neq s\rangle$$
$$= |x\neq s\rangle$$

$\boxed{G}$

$$G = -I + 2|\alpha\rangle\langle\alpha| \quad \text{where} \quad |\alpha\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle$$

$$G|\gamma\rangle = \begin{cases} |\gamma\rangle & \text{if } |\gamma\rangle = |\alpha\rangle \\ -|\gamma\rangle & \text{if } \langle\gamma|\alpha\rangle = 0 \end{cases}$$

Grovers Alg:



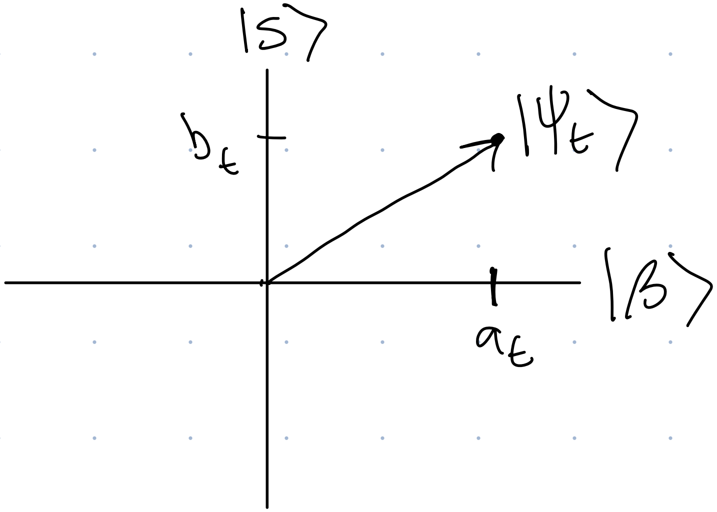
State is 2N dimensional <u>BUT</u> throughout alg the state only lives in 2 of those dimensions

In particular:

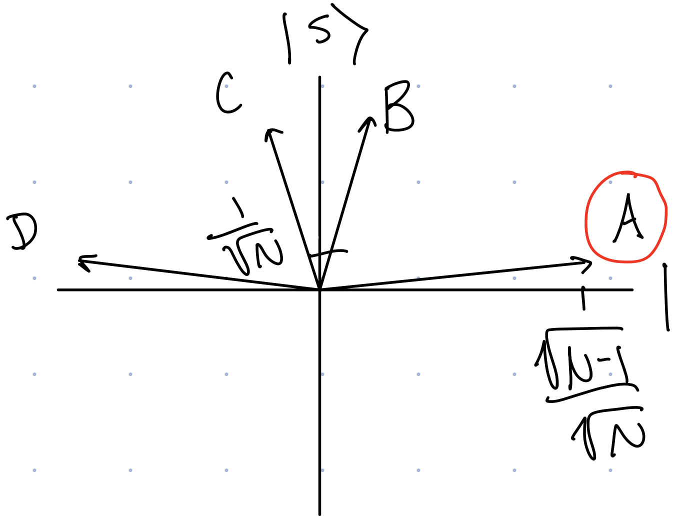
$$|\psi_t\rangle = a_t|\beta\rangle + b_t|s\rangle \qquad |\beta\rangle = \frac{1}{\sqrt{N-1}}\sum_{i\neq s}|i\rangle$$

$$a_t, b_t \in \mathbb{R}$$

We can express $|\psi_t\rangle = a_t|\beta\rangle + b_t|s\rangle$ as a vector in 2D:



Where is $|\alpha\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$? (assume N is big)



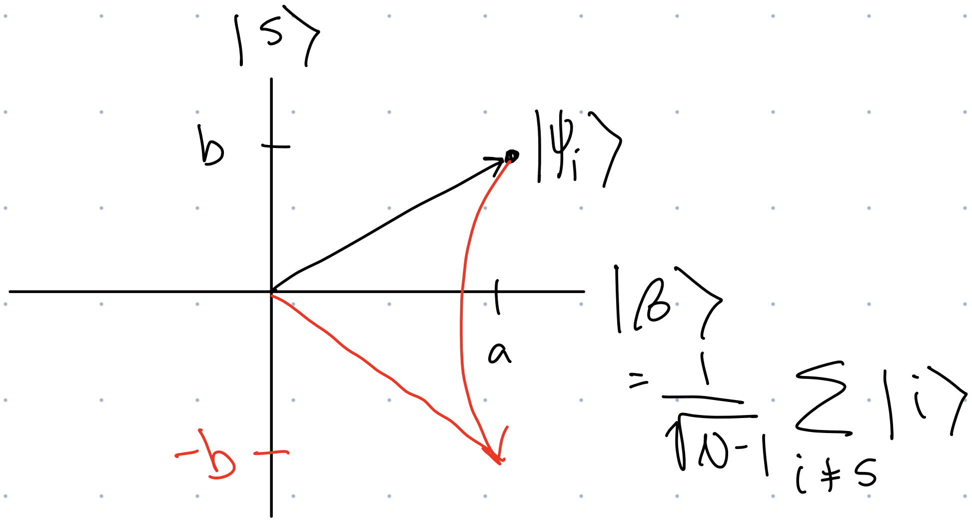
$|\beta\rangle = \frac{1}{\sqrt{N-1}}\sum_{i\neq s}|i\rangle$

$$|\alpha\rangle = a\left(\frac{1}{\sqrt{N-1}}\sum_{i\neq s}|i\rangle\right) + b|s\rangle$$

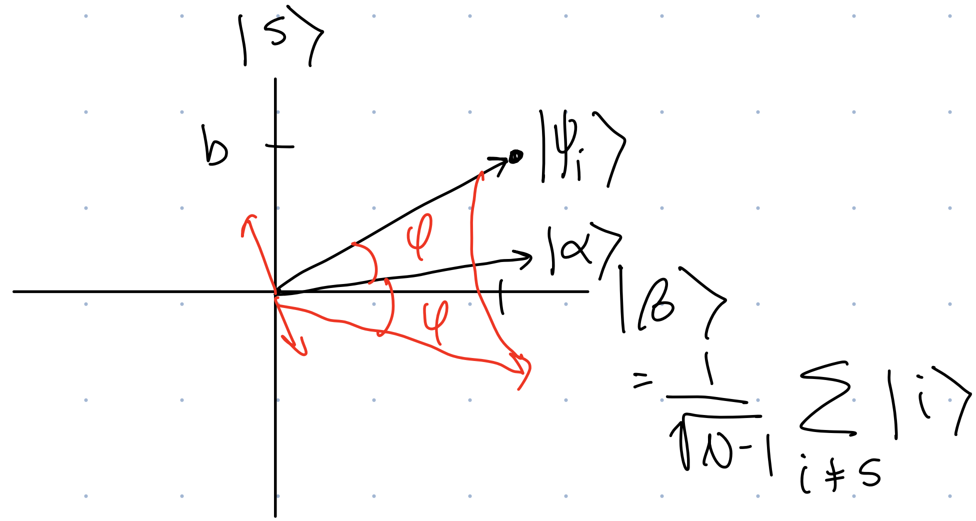
$a = \frac{\sqrt{N-1}}{\sqrt{N}}$

$\frac{1}{\sqrt{N}}$

Effect of $U_f' = I - 2|s\rangle\langle s|$

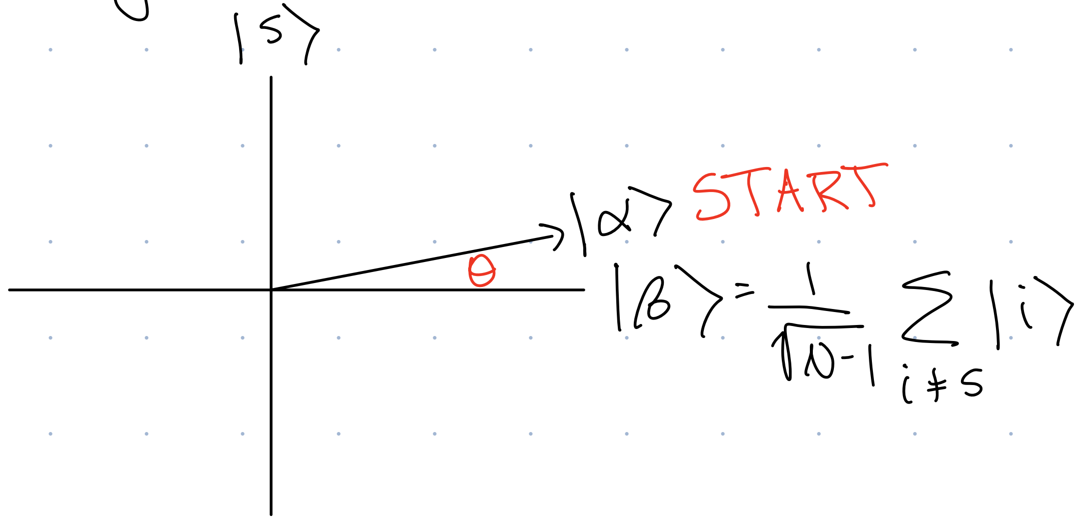


Effect of $G = -I + 2|\alpha\rangle\langle\alpha|$



How many iterations before state becomes $|s\rangle$?

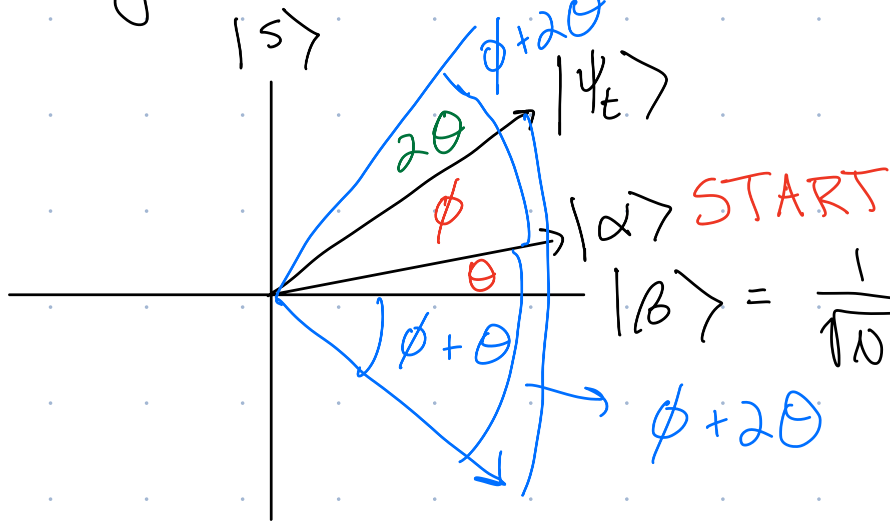
$(U_f', G, U_f', G, \ldots)$



$\tan^{-1}(x) \approx x$

for $x \ll 1$

How many iterations before state becomes $|s\rangle$?



$$|\beta\rangle = \frac{1}{\sqrt{N-1}} \sum_{i \neq s} |i\rangle$$

$$n \cdot 2\theta = \frac{\pi}{2} = (90^\circ)$$

$$\tan^{-1}(\tan(\theta)) = \frac{\frac{1}{\sqrt{N}}}{\frac{\sqrt{N-1}}{\sqrt{N}}} = \tan^{-1}\left(\frac{1}{\sqrt{N-1}}\right)$$

$$\theta \approx \frac{1}{\sqrt{N-1}}$$

$$n = \frac{\pi}{4}\sqrt{N-1}$$

$$n = O(\sqrt{N})$$

Note

Application

ex: