FACTORING

Learning Goals

- Analyze a multi-qubit algorithm
- Become familiar with Shor's factoring algorithm (most famous q. alg)

Announcements

- Proj Update:
- No OH Tuesday
- Exam: all but QC4 (11/13 Thurs. in class) 2 3x5" notecards

- Need help researching your project?
  → STEM librarian drop-in hours:
  - Wed 2-4
  - Thurs. 12-2

  Q-Center
- TidyTuesdays - opensource worldwide data analysis
  → Tuesday in Nov. 3:30-5 in Q-Center

- Review notes (PS7) (3 min, 1-2 pts to share)
- Small group sharing (5 min)
- Larger group reflecting on something you heard. (25 min)

The fields of quantum computing and computer science are engaged in discussions perceived inclusive language vs. perceived exclusionary language. We are asking questions like: can language choices foster inclusivity? Are inclusive language choices mere virtue signalling? How do we draw the line between acceptable and unacceptable language as social norms and language changes? Is it important for scientific communities to engage in reflection and discussion of language use, or should scientific communities focus their attention on scientific discovery?

Name ____________________________

CS333 - Long In-Class Exam 2

Put your answers entirely in the boxes corresponding to that question. If you need additional space, put a note *within* the corresponding box saying that the work continues on scratch paper, and clearly label any additional pages you submit with the problem number and your name.

This exam should be completed on your own, with at most two 3 inch x 5 inch note-cards.

Please write and sign the honor code in the box. (I have neither given nor received unauthorized aid on this assessment.)

Possibly helpful things:

- The geometric series formula:

$$\sum_{m=0}^{t-1} a^m = \begin{cases} t & \text{if } a = 1 \\ \frac{1-a^t}{1-a} & \text{else} \end{cases} \tag{1}$$

- QFT_N, which is QFT acting on an N-dimensional state, transforms the standard basis state $|w\rangle$ as

$$|w\rangle \to \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i x w/N} |x\rangle, \tag{2}$$

- The gate Y acts as:

$$|0\rangle \to i|1\rangle \tag{3}$$
$$|1\rangle \to -i|0\rangle \tag{4}$$

Exit Tickets

- Non-repetetive function? → Next algorithm
- Other Similar Problems? → PS9
- Largest number factored using Shor's alg*

A) 15 — 3×5 B) 35 — 5×7 C) 91 — 9×13 D) 221 — 13×17

* without shortcuts that are not scalable [Monz et al 2016, Smolin et al]

- Classical Part on Exams?

Exit Tickets

$$|\psi_3\rangle = \sqrt{\frac{r}{N}} \sum_{m=0}^{\frac{N}{r}-1} |b^* + mr\rangle \quad \text{(after partial meas. collapse)}$$

$$|\psi_4\rangle = QFT\,|\psi_3\rangle = \sqrt{\frac{r}{N}} \sum_{m=0}^{\frac{N}{r}-1} QFT\,|b^* + mr\rangle$$

$$= \sqrt{\frac{r}{N}} \sum_{m=0}^{\frac{N}{r}-1} \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i y (b^* + mr)/N} |y\rangle$$

① Switch order of summation

$$\sum_{y} \Big(\qquad \Big) |y\rangle$$

② Geometric Series

QCS

QC3

Public Key Crypto (RSA)

0 Secret message
$m \in \{0,1\}^n$

Eve has access to: $\bar{m}$, K_{pub}, f, g $\rightarrow m$

No direct access: m, K_{pri}

If she could solve a hard math problem $\rightarrow m$

factoring a large number

$$\underbrace{6\,7\,0\,1\,1\,2\,8\,7\,3\,6\ldots}_{617 \text{ digits}} = a \times b$$

?

\$200K prize

\$100K for 308 digit

$\checkmark$ 250 digit

Quantum Computes Can Efficiently Factor

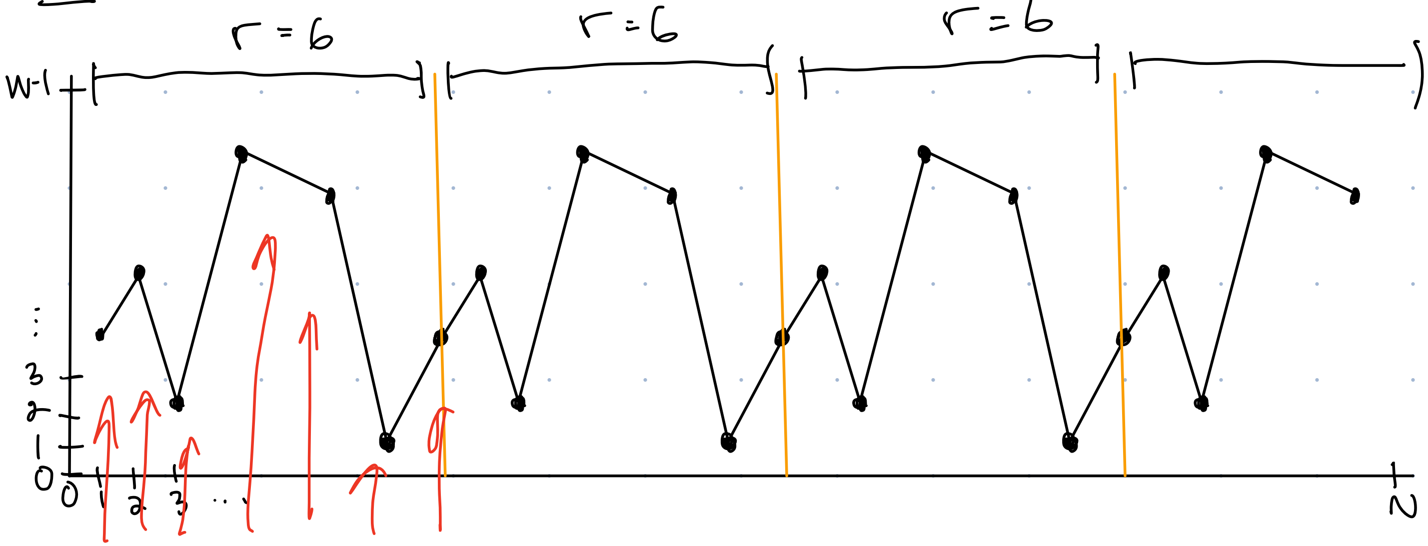
Factoring reduces (via number theory) to period finding:

Period Finding Problem

Input: Query access to $f: \{0,1,2,\dots,N-1\} \to \{0,1,2,\dots W-1\}$

- f has a period r (unknown to you)
- no repeated values within a period
- $N > r^2$ (many repeats)

ex:



Output: r

What is the classical query complexity of ~~factoring~~ period finding?

A) $O(\sqrt{r})$ B) $O(r)$ C) $O(r^2)$ D) $O(N)$

Bits to Digits

Classically, we code using base 10, not binary. We'll do same:

$011 \longleftrightarrow 3$

$|011\rangle \longleftrightarrow |3\rangle$

$|0011\rangle$

2^4

16-dimensional

Standard Basis States:

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$\updownarrow \quad \updownarrow \quad \updownarrow$

$\underbrace{\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \dots |N-1\rangle\}}_{\text{N-dimensional system}}$

$\sum_{i \in \{0,1\}^m} |i\rangle \longleftrightarrow \sum_{j=0}^{2^m-1} |j\rangle$

Ambiguity:

$|3\rangle \to |11\rangle$

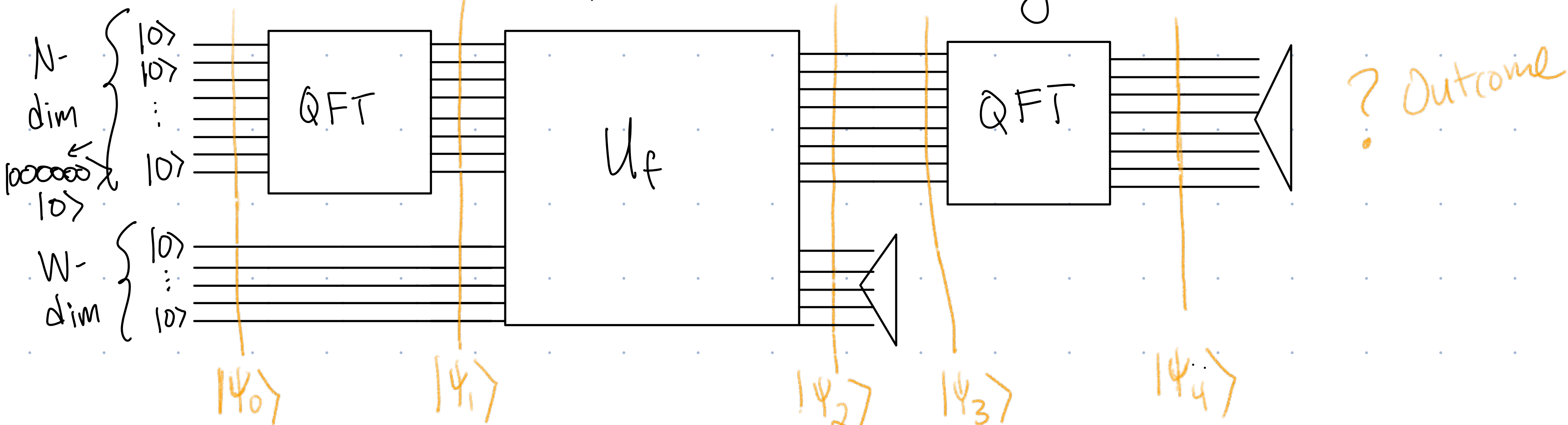
$|3\rangle \to |011\rangle$

$|3\rangle \to |0000011\rangle$

If $|3\rangle$ is an N-dimensional state, how many qubits are in the system?

A) $\lceil \log_2 3 \rceil$ B) 3 C) $\lceil \log_2 N \rceil$ D) N

Quantum Circuit for Period Finding



QFT: Quantum Forier Transform

$$|x\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i x y/N} |y\rangle$$

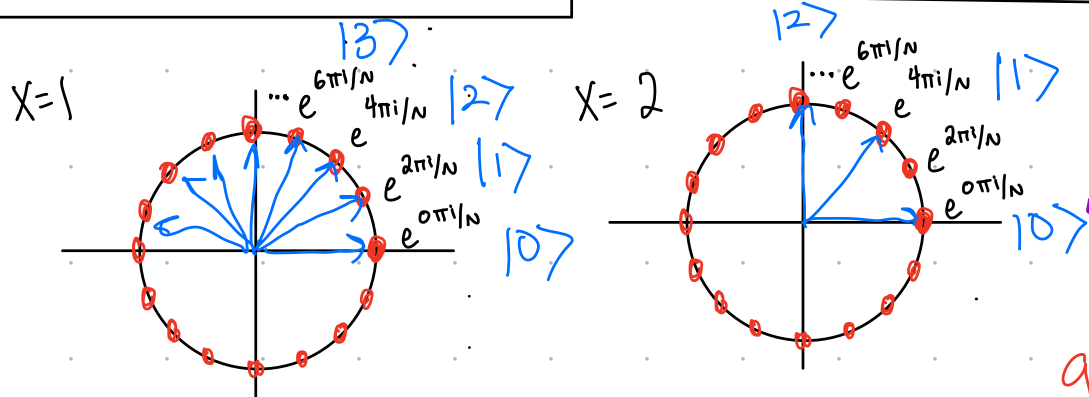
N-dim
standard basis state

U_f:

$$|x\rangle|y\rangle = |x\rangle|f(x)+y \bmod W\rangle$$

N-dim W-dim
s.b. states

Phase of each s.b. state in superposition

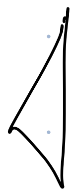


$\boxed{QFT|0\rangle = ?}$

$$= \sum_{i=0}^{N-1} \frac{1}{\sqrt{N}} |y\rangle$$

go/fourier-transform

Reminder: $U\left(\sum_i a_i|i\rangle\right) = \sum_i a_i U|i\rangle$

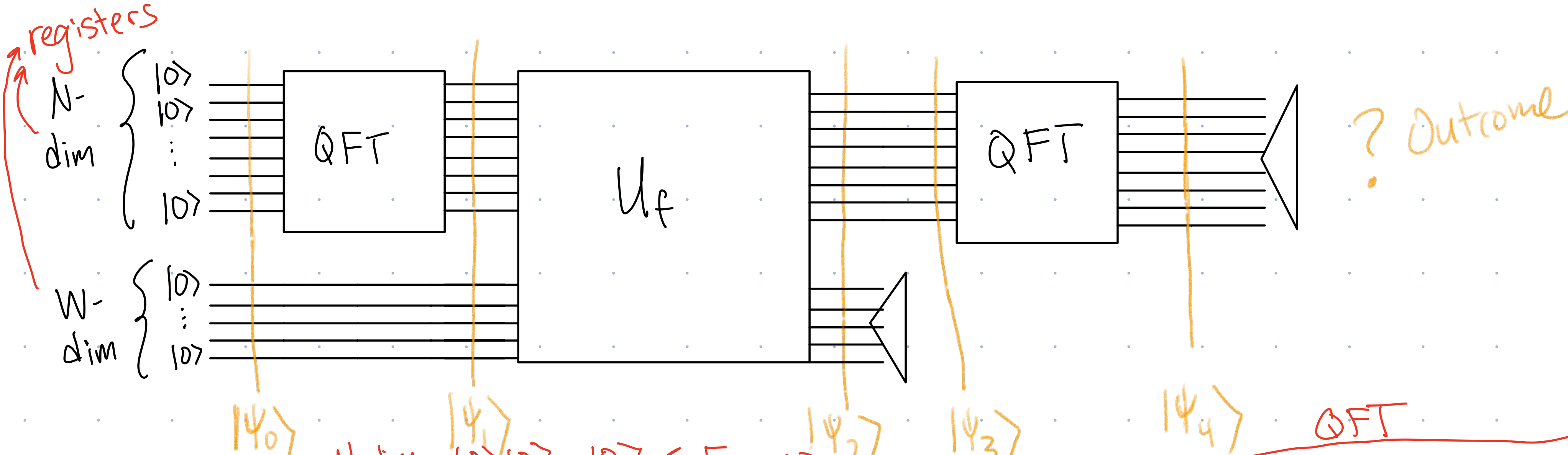
 = Standard basis measurement

- Measure entire state:

$$|\psi\rangle=\sum_{x=0}^{N-1} a_x|x\rangle \Longrightarrow \text{Prob of outcome } |x^*\rangle \text{ is } |a_{x^*}|^2$$

- Partial measurement (B system)
    - Factor standard basis states of measured register
    - Collapse + renormalize

Group Excercise: Analyze!



N-dim $\quad |0\rangle|0\rangle\cdots|0\rangle \leftarrow \lceil \log_2 N\rceil$ qubits

W-dim

$$|\psi_0\rangle = |0\rangle|0\rangle$$

QFT

$$|x\rangle \rightarrow \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i x y/N}|y\rangle$$

$$|\psi_1\rangle = (QFT\otimes I)|0\rangle|0\rangle = (QFT|0\rangle)\otimes|0\rangle$$

$$= \left(\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i\cdot 0\, y/N}|y\rangle\right)\otimes|0\rangle = \left(\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}|y\rangle\right)\otimes|0\rangle$$

U_f

$$|x\rangle|y\rangle = |x\rangle|(f(x)+y) \bmod W\rangle$$

$$|\psi_2\rangle = U_f\left[\left(\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}|y\rangle\right)\otimes|0\rangle\right]$$

$$= \frac{U_f}{\sqrt{N}}\left[(|0\rangle+|1\rangle+|2\rangle+\cdots|N-1\rangle)|0\rangle\right]$$

$$= U_f \frac{1}{\sqrt{N}} \left(|0\rangle|0\rangle + |1\rangle|0\rangle + |2\rangle|0\rangle + \cdots |N-1\rangle|0\rangle \right)$$

$$= U_f \frac{1}{\sqrt{N}} \left(\sum_{y=0}^{N-1} |y\rangle|0\rangle \right)$$

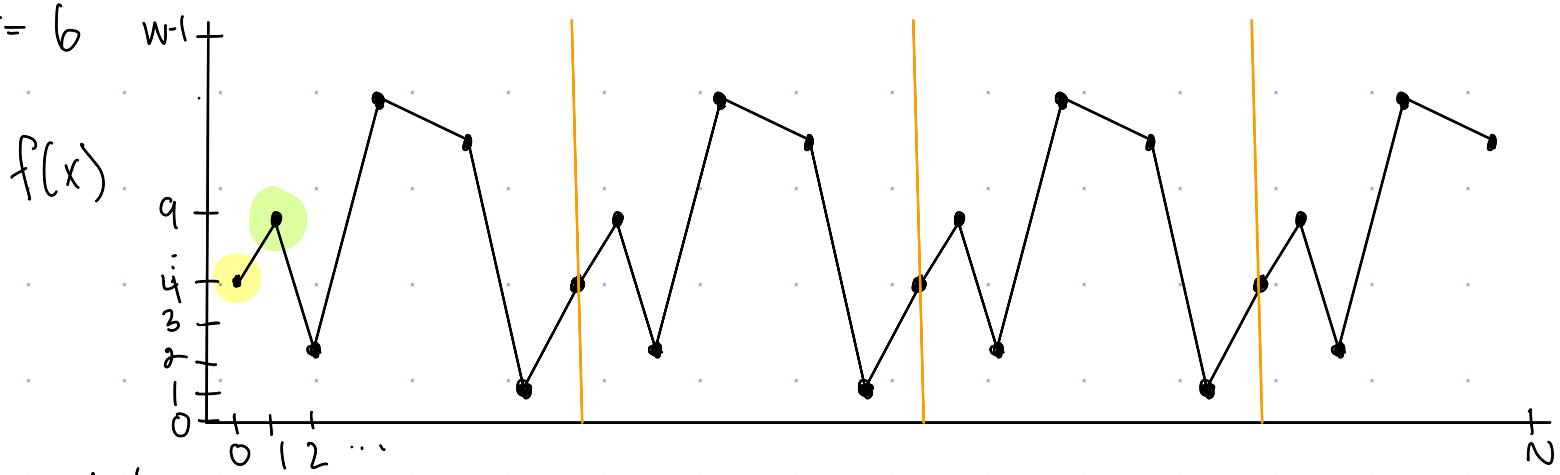
$$= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} U_f |x\rangle|0\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle|0 + f(x) \bmod w\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle|f(x)\rangle$$

ex:

$r = 6$



$$\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle |f(x)\rangle =$$

x

$$\frac{1}{\sqrt{N}} \Big(|0\rangle_A |4\rangle_B + |1\rangle_A |9\rangle_B + |2\rangle_A |2\rangle_B + |3\rangle|13\rangle + |4\rangle|12\rangle + |5\rangle|1\rangle +$$
$$|6\rangle|4\rangle + |7\rangle|9\rangle + |8\rangle|2\rangle + |9\rangle|13\rangle + |10\rangle|12\rangle + |11\rangle|1\rangle +$$
$$|12\rangle|4\rangle + |13\rangle|9\rangle + |14\rangle|2\rangle + |15\rangle|13\rangle + |16\rangle|12\rangle + |17\rangle|1\rangle + \cdots$$

measure $|9\rangle$ collapse

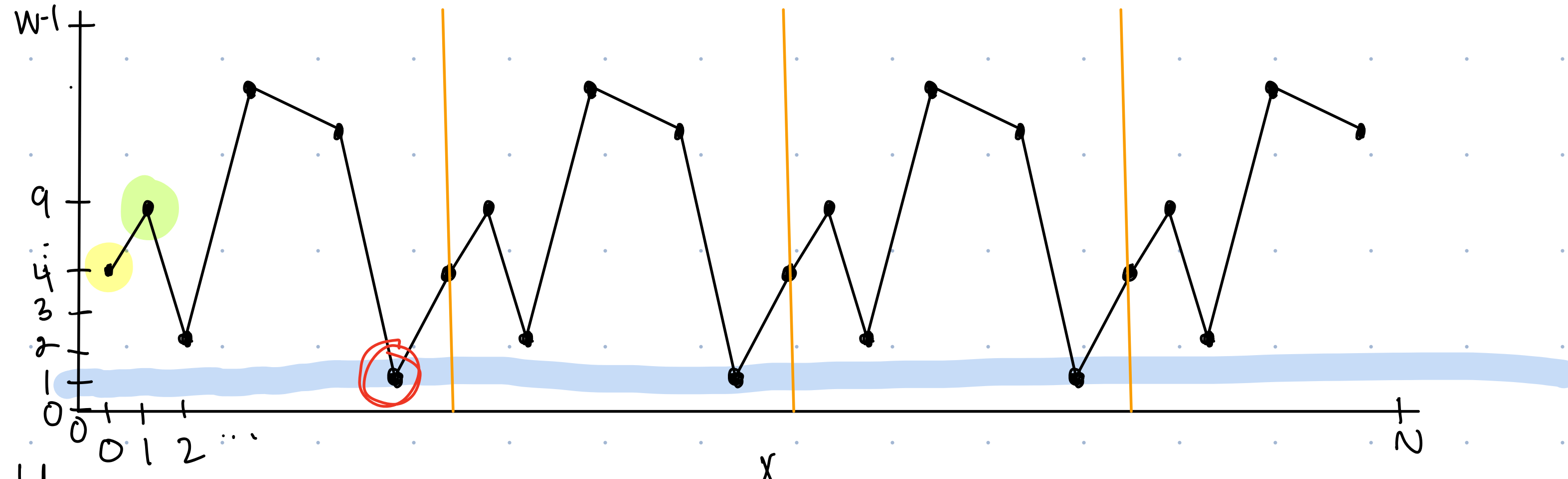
measure $|13\rangle$ collapse

$$\frac{1}{\sqrt{p(9)}} \left(\frac{1}{\sqrt{N}} |1\rangle + \frac{1}{\sqrt{N}} |7\rangle + \frac{1}{\sqrt{N}} |13\rangle + \cdots \right) |9\rangle$$

$$\frac{1}{\sqrt{p(13)}} \left(\frac{1}{\sqrt{N}} |3\rangle + \frac{1}{\sqrt{N}} |9\rangle + \frac{1}{\sqrt{N}} |15\rangle + \cdots \right) |13\rangle$$

HOW TO REPRESENT THE COLLAPSED STATE?

$r = 6$



After U_f:

$$\frac{1}{\sqrt{N}}\Big(|0\rangle_A|4\rangle_B + |1\rangle_A|9\rangle_B + |2\rangle_A|2\rangle_B + |3\rangle|13\rangle + |4\rangle|12\rangle + |5\rangle|1\rangle +$$
$$|6\rangle|4\rangle + |7\rangle|9\rangle + |8\rangle|2\rangle + |9\rangle|13\rangle + |10\rangle|12\rangle + |11\rangle|1\rangle +$$
$$|12\rangle|4\rangle + |13\rangle|9\rangle + |14\rangle|2\rangle + |15\rangle|13\rangle + |16\rangle|12\rangle + |17\rangle|1\rangle + \cdots$$

If measure outcome $|y\rangle$ in 2nd register, let b^* be the value such that:

- $f(b^*) = y$
- b^* is in the first period.

$\Big\}$ define b^*

If outcome is $|1\rangle$, what is b^* for above f?

A) 0 B) 1 C) 5 D) No b^* exists.

$$\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle |f(x)\rangle =$$

$$\frac{1}{\sqrt{N}} \Big(|0\rangle_A |4\rangle_B + |1\rangle_A |9\rangle_B + |2\rangle_A |2\rangle_B + |3\rangle|13\rangle + |4\rangle|12\rangle + |5\rangle|1\rangle +$$
$$|6\rangle|4\rangle + |7\rangle|9\rangle + |8\rangle|2\rangle + |9\rangle|13\rangle + |10\rangle|12\rangle + |11\rangle|1\rangle +$$
$$|12\rangle|4\rangle + |13\rangle|9\rangle + |14\rangle|2\rangle + |15\rangle|13\rangle + |16\rangle|12\rangle + |17\rangle|1\rangle + \cdots$$

measure $|9\rangle$ collapse

measure $|13\rangle$ collapse

b^* $\quad b^*+r \quad b^*+2r$

$$\frac{1}{\sqrt{p(9)}} \left(\frac{1}{\sqrt{N}} |1\rangle + \frac{1}{\sqrt{N}} |7\rangle + \frac{1}{\sqrt{N}} |13\rangle + \cdots \right) |9\rangle$$

$$\frac{1}{\sqrt{p(13)}} \left(\frac{1}{\sqrt{N}} |3\rangle + \frac{1}{\sqrt{N}} |9\rangle + \frac{1}{\sqrt{N}} |15\rangle + \cdots \right) |13\rangle$$

$$\frac{1}{\sqrt{p(f(b^*))}} \frac{1}{\sqrt{N}} \sum_{m=0}^{\frac{N}{r}-1} |b^* + m\underset{\text{period}}{r}\rangle |\cancel{f(b^*)}\rangle$$

ignore

$\frac{N}{r}$ = # of repeats of periods

$$\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle|f(x)\rangle =$$

$$\frac{1}{\sqrt{N}}\Big(|0\rangle_A|4\rangle_B + |1\rangle_A|9\rangle_B + |2\rangle_A|2\rangle_B + |3\rangle|13\rangle + |4\rangle|12\rangle + |5\rangle|1\rangle +$$
$$|6\rangle|4\rangle + |7\rangle|9\rangle + |8\rangle|2\rangle + |9\rangle|13\rangle + |10\rangle|12\rangle + |11\rangle|1\rangle +$$
$$|12\rangle|4\rangle + |13\rangle|9\rangle + |14\rangle|2\rangle + |15\rangle|13\rangle + |16\rangle|12\rangle + |17\rangle|1\rangle + \cdots$$

measure $|9\rangle$ collapse

measure $|13\rangle$ collapse

$$\frac{1}{\sqrt{p(9)}}\left(\frac{1}{\sqrt{N}}|1\rangle + \frac{1}{\sqrt{N}}|7\rangle + \frac{1}{\sqrt{N}}|13\rangle + \cdots\right)|9\rangle$$

$$\frac{1}{\sqrt{p(13)}}\left(\frac{1}{\sqrt{N}}|3\rangle + \frac{1}{\sqrt{N}}|9\rangle + \frac{1}{\sqrt{N}}|15\rangle + 0 \cdots 0\right)|13\rangle$$

Generally: $$\frac{1}{\sqrt{p(f(b^*))}}\,\frac{1}{\sqrt{N}}\sum_{m=0}^{N/r-1}|b^*+mr\rangle\cancel{|f(b^*)\rangle}$$

Add abs squared of amplitudes in this part

$$= \left|\frac{1}{\sqrt{N}}\right|^2 + \left|\frac{1}{\sqrt{N}}\right|^2 + \cdots \left|\frac{1}{\sqrt{N}}\right|^2 = (\#)\cdot\frac{1}{N} = \frac{N}{r}\cdot\frac{1}{N} = \frac{1}{r}$$

Suppose measure outcome $|f(b^*)\rangle$

$$|\psi_3\rangle = \frac{1}{\sqrt{\frac{1}{r}}} \frac{1}{\sqrt{N}} \sum_{m=0}^{\frac{N}{r}-1} |b^* + mr\rangle$$

$$= \sqrt{\frac{r}{N}} \sum_{m=0}^{\frac{N}{r}-1} |b^* + mr\rangle$$

$$|\psi_4\rangle = \sqrt{\frac{r}{N}} \sum_{m=0}^{\frac{N}{r}-1} \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i y(b^*+mr)/N} |y\rangle$$

$$\overbrace{\omega|0\rangle + \omega|1\rangle + \omega|2\rangle + \dots \omega|N-1\rangle}^{m=0} + \overbrace{\xi|0\rangle + \xi|1\rangle + \dots + \xi|N-1\rangle}^{m=1} + \overbrace{\text{———}}^{m+2}$$

Switch order of summation:

$$= \sqrt{\frac{r}{N}} \sqrt{\frac{1}{N}} \sum_{y=0}^{N-1} \left(\sum_{m=0}^{\frac{N}{r}-1} \underbrace{\boxed{e^{2\pi i y(b^*+mr)/N}}}_{e^{2\pi i y b^*/N}\, e^{2\pi i y mr/N}} \right) |y\rangle$$

Factor out $e^{2\pi i y b^*/N}$

$$|\psi_4\rangle = \frac{\sqrt{r}}{N} \sum_{y=0}^{N-1} \left(e^{2\pi i y b^*/N} \sum_{m=0}^{\frac{N}{r}-1} e^{2\pi i y m r/N} \right) |y\rangle$$

$$= \sum_{y=0}^{N-1} \frac{\sqrt{r}}{N} e^{2\pi i b^* y/N} \sum_{m=0}^{\frac{N}{r}-1} \left(e^{2\pi i y r/N} \right)^m |y\rangle$$

Pull m out of exponent

Prob. of outcome y: $\left| \frac{\sqrt{r}}{N} e^{2\pi i b^* y/N} \boxed{\sum_{m=0}^{\frac{N}{r}-1} \left(e^{2\pi i y r/N} \right)^m} \right|^2$

Geometric Series: $\sum_{m=0}^{t-1} a^m$

In our case:

- $a = e^{2\pi i y r/N}$
- $t = N/r$

$$\sum_{m=0}^{t-1} a^m = \begin{cases} t & \text{if } a=1 \\ \dfrac{1-a^t}{1-a} & \text{if } a\neq 1 \end{cases}$$

Case 1

$a \neq 1 \leftrightarrow e^{2\pi i y r/N} \neq 1$

$$\sum_{m=0}^{\frac{N}{r}-1} \left(e^{2\pi i y r/N}\right)^m = \frac{1-e^{\frac{2\pi i y r}{N}\cdot\frac{N}{r}}}{1-e^{2\pi i y r/N}}$$

$$= \frac{1-\cancelto{1}{e^{2\pi i y}}}{1-e^{2\pi i y r/N}}$$

$$=0$$

Case 2

$a = 1 \leftrightarrow e^{2\pi i y r/N} = 1$

$\leftrightarrow \frac{yr}{N} = k \in \mathbb{Z}$

$\leftrightarrow y = \frac{Nr}{r}$ for $k \in \mathbb{Z}$

$$\sum_{m=0}^{\frac{N}{r}-1} \left(e^{2\pi i y r/N}\right)^m = \frac{N}{r}$$

Prob. of outcome y: $\left| \frac{\sqrt{r}}{N} e^{2\pi i b^* y/N} \boxed{\sum_{m=0}^{\frac{N}{r}-1} \left(e^{2\pi i y r/N}\right)^m} \right|^2$

Plugging in:

Case 1

$$\left| \frac{\sqrt{r}}{N} e^{2\pi i b^* y/N} \cdot 0 \right|^2 = 0$$

Case 2

$$\left| \frac{\sqrt{r}}{N} e^{2\pi i b^* y/N} \frac{N}{r} \right|^2$$

$$= \left| \cancel{e^{2\pi i b^* y/N}} \right|^2 \frac{1}{r}$$

1

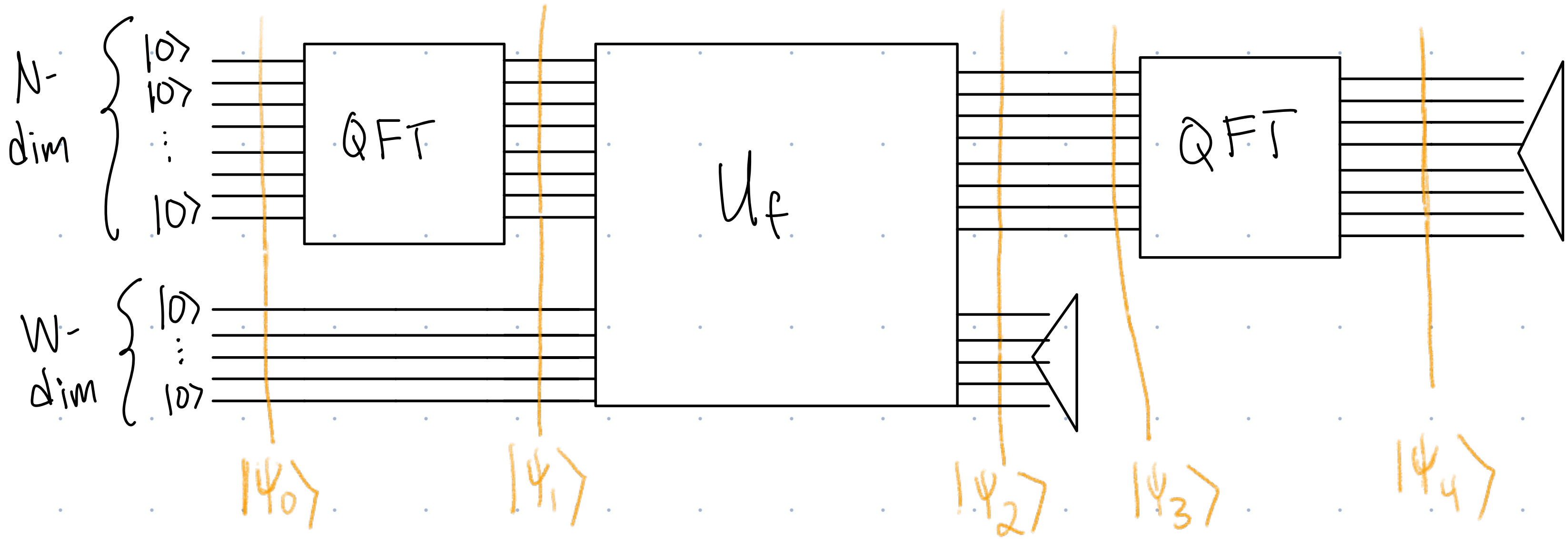
$$= \frac{1}{r}$$

End result: Final measurement of first register will always result in a multiple of $\frac{N}{r}$

$$\Rightarrow \quad y = \frac{kN}{r} \quad \text{for } k \in \mathbb{Z}$$

Period Finding Algorithm (Shor's Algorithm)

① Run 2 times:



$z_1 = \frac{Nk_1}{r}$

$z_2 = \frac{Nk_2}{r}$

② Continued Fractions $y_1 \to \frac{s_1}{j_1} = \frac{k_1 N}{r}$ $y_2 \to \frac{s_2}{j_2} = \frac{k_2 N}{r}$

(i) j_1 or j_2 will be r (ii) j_1, j_2 are factors of $r \to$ l.c.m.$(j_1, j_2) \Rightarrow r$

③ Check $f(0) \overset{?}{=} f(r)$

Successful with prob. $\geq 2/3$

Query Compexity of Period Finding

Classical: $O(\sqrt{r})$

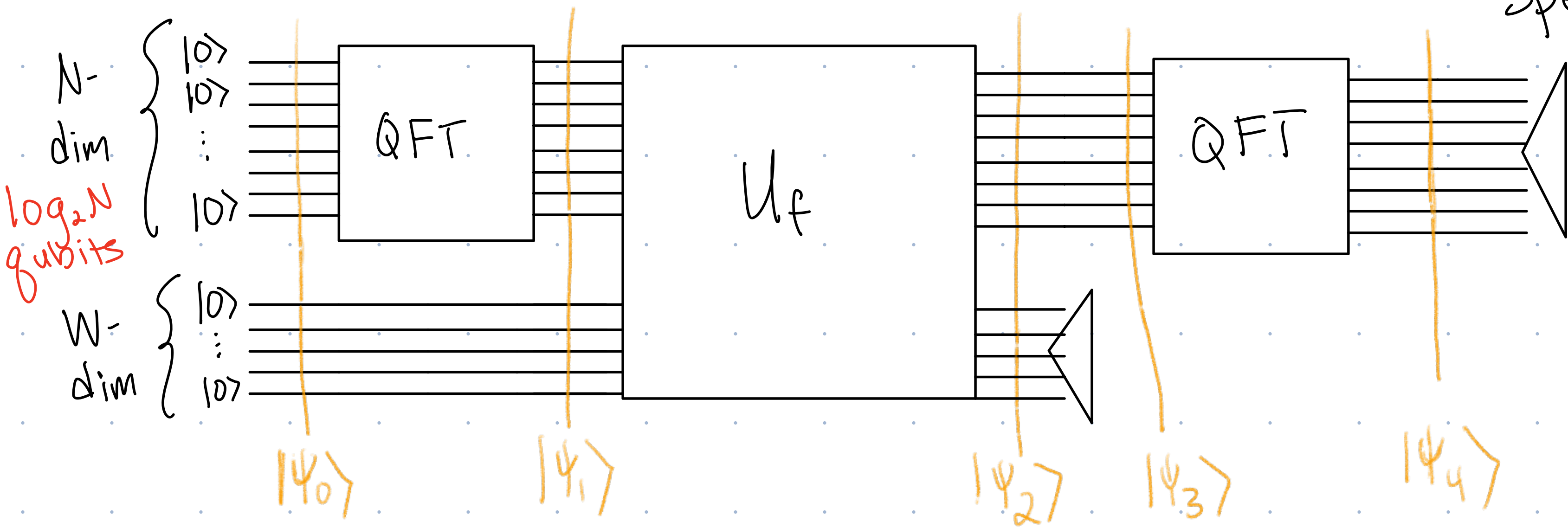
Quantum: $O(1)$

Time Complexity of Factoring

Quantum: $O(\log_2 N)$

Nearly Exponential Speedup!

Circuit used to factor N.



$O(\log_2^2 N)$ $O(\log_2 N)$ $O(\log_2^2 N)$ $\rightarrow$ $O(\log_2^2 N)$

Classical: $e^{O(\log_2^{1/3} N)}$ Number Field Sieve

Pesky Detail:

We looked at prob of: $y = \frac{kN}{r}$

But if $\frac{N}{r} \notin \mathbb{N}$, y is a fraction... see pset.