Error Correction

Learning Goals

- Describe errors in quantum/classical systems
- Describe repetition codes + why they work
- Analyze how a repetition code corrects errors

Announcements

- Mon 12:30-1:30 What I did this summer + Pizza
- No class Tues: Work on Project!

Exit Tickets

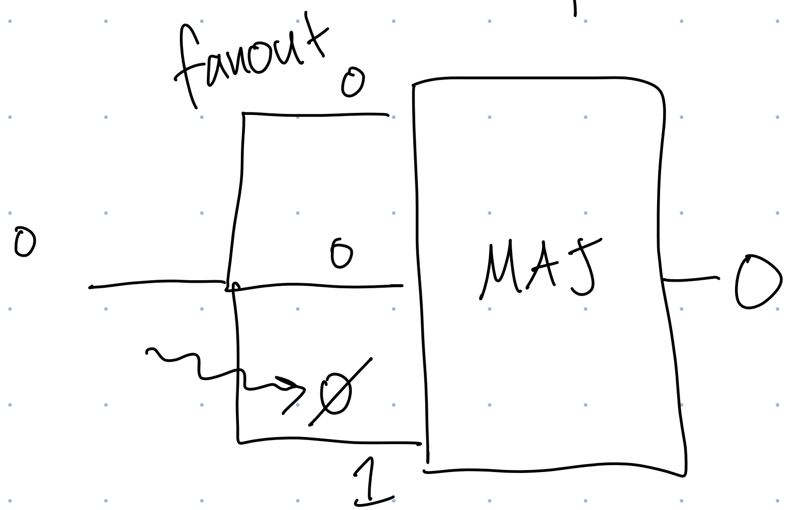
- How to determine P_i's?
- P_i form?

If Q. Computers are so great, why haven't we built one?

Errors - unwanted gates/measurements

- Classical Computers
  - cosmic rays — "soft errors"
    - ↳ 1 bit flip / 4 GB per day

Solution: <u>Repetition Code</u>

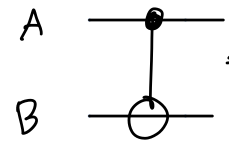


- Sources of Quantum Errors
  - control lasers not perfect (shape, frequency, focus, intensity)
  - imperfect vacuum
  - Stray magnetic, electric fields
  - Heat (anomalous heating)

Why won't repetition code work for quantum?

A) MAJ not reversible
B) Doesn't correct phase errors (Z)
C) No fanout because can't copy quantum states
D) There are uncountably infinitely many errors to deal with

Shor's Repetition Code

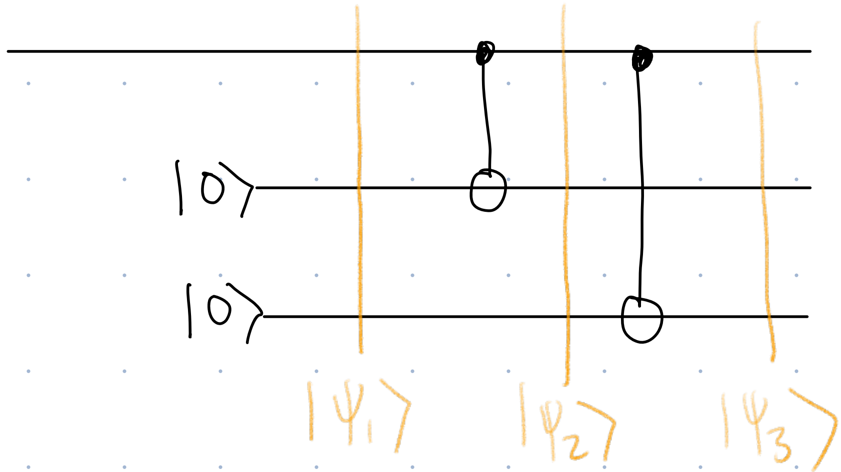


= CNOT

If $A = |1\rangle$, $\rightarrow X B$

Encoding:

$|\psi\rangle = a|0\rangle + b|1\rangle$



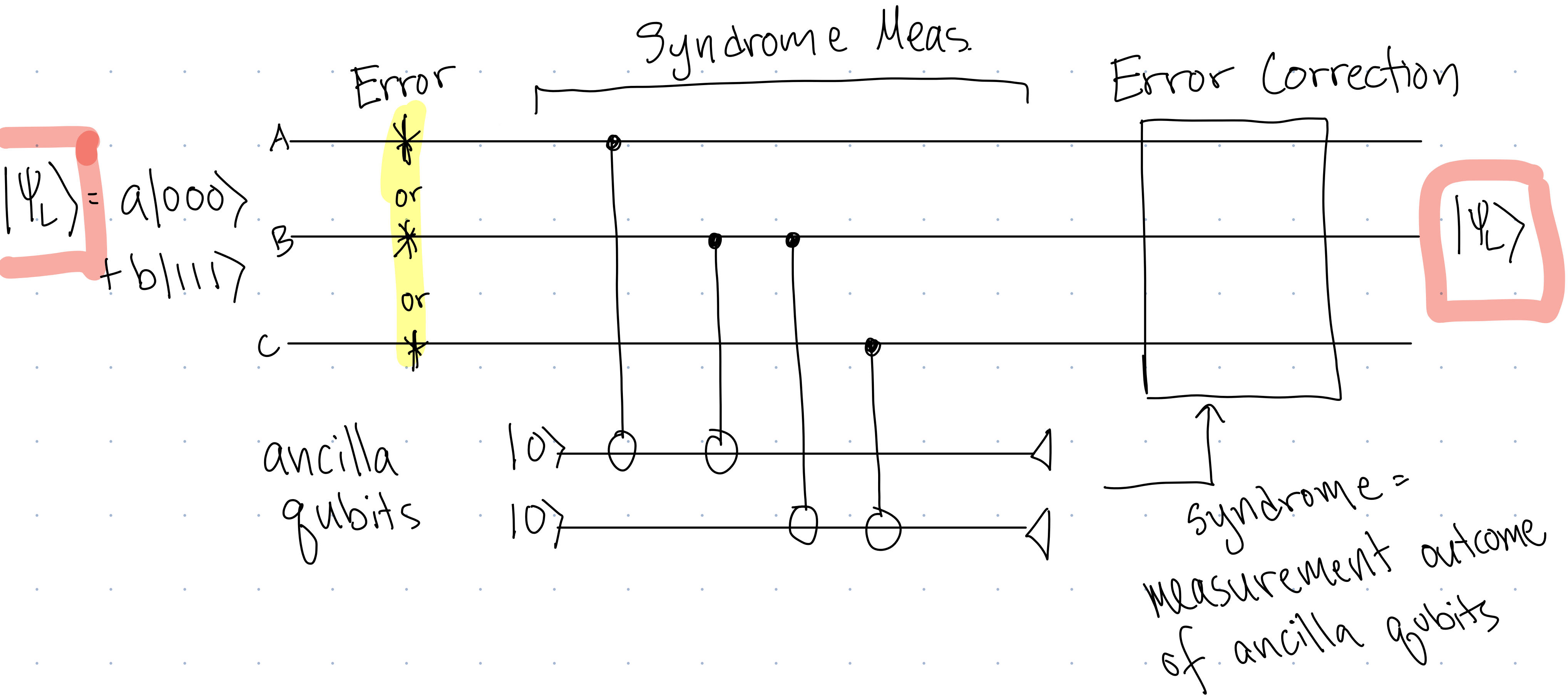
original state to protect from errors. Don't know a, b.

$$|\psi_1\rangle = \left(a|0\rangle + b|1\rangle\right)|0\rangle|0\rangle = a|000\rangle + b|100\rangle$$

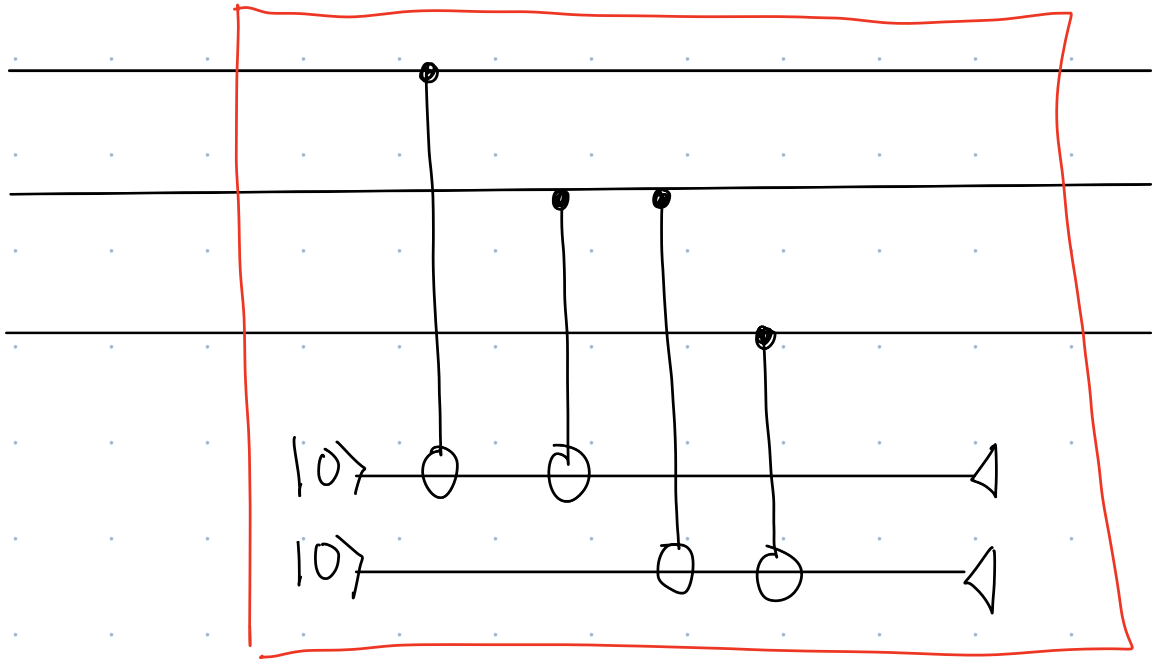
$$|\psi_2\rangle = a\,CNOT_{12}|000\rangle + b\,CNOT_{12}|100\rangle$$
$$= a|000\rangle + b|110\rangle$$

$$|\psi_3\rangle = a\,CNOT_{13}|000\rangle + b\,CNOT_{13}|110\rangle$$
$$= a|000\rangle + b|111\rangle$$

Error Correction Circuit

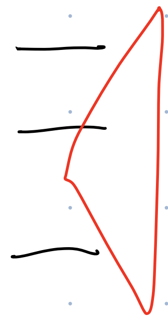


Syndrome Meas.



Projective Meas.

$\approx$



Projective Measurement

Described by a set of projectors $M = \{P_0, P_1, P_2, \dots\}$

$\uparrow$ outcomes

ex: $P_0 = |\phi_1\rangle\langle\phi_1| + |\phi_4\rangle\langle\phi_4|$

$P_1 = |\phi_3\rangle\langle\phi_3|$

$P_2 = |\phi_2\rangle\langle\phi_2| + |\phi_5\rangle\langle\phi_5| + |\phi_6\rangle\langle\phi_6|$

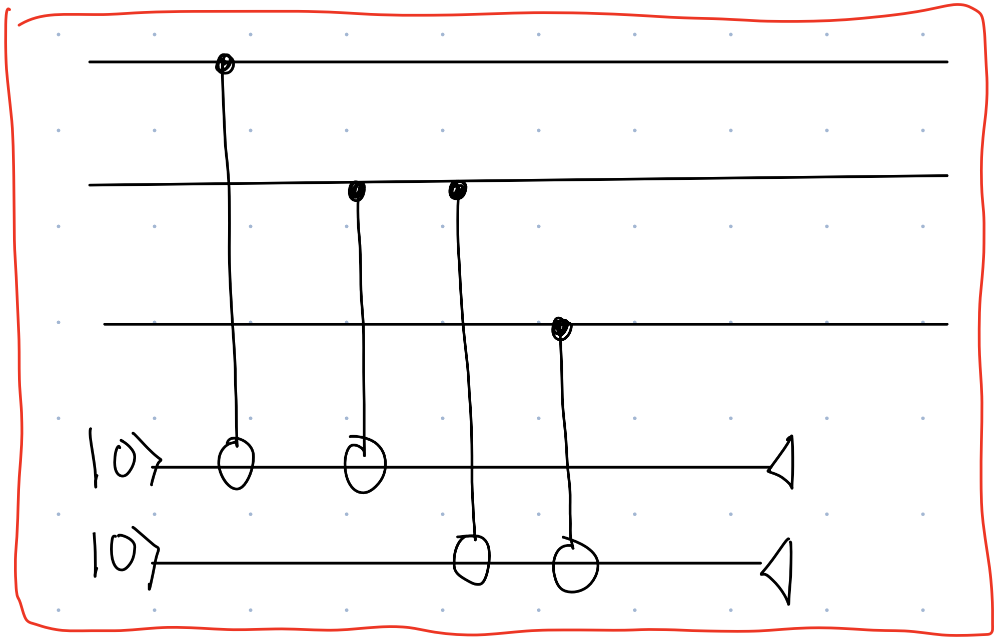
$\boxed{|\phi\cancel{\rangle\langle}\phi| \text{ write as } |\phi\rangle\langle\phi|}$

- $|\phi_i\rangle$ should be N-dimensional, where N is dim. of system being measured
- Technical $\sum P_i = I$

If measure $|\psi\rangle$

→ Probability of outcome P_i: $\langle\psi|P_i|\psi\rangle$

→ If outcome i: collapse to $\dfrac{P_i|\psi\rangle}{\sqrt{Pr(i) = \langle\psi|P_i|\psi\rangle}}$



$$\approx \quad M = \{ \underbrace{|000\rangle\langle 000| + |111\rangle\langle 111|}_{P_0},\ \underbrace{|001\rangle\langle 001| + |110\rangle\langle 110|}_{P_1},\ \underbrace{|010\rangle\langle 010| + |101\rangle\langle 101|}_{P_2},\ \underbrace{|100\rangle\langle 100| + |011\rangle\langle 011|}_{P_3} \}$$

$$\langle \psi | P | \psi \rangle$$

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle$ with M, which outcomes are possible?

$\boxed{\text{A) } P_0 \quad \text{B) } P_1}$

C) P_2

D) P_3

$$\overbrace{\left(\langle\psi|\right)}^{\langle\psi|}\ \overbrace{\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)}^{P_0}\ \overbrace{\left(\frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)}^{|\psi\rangle}$$

$$= \langle\psi|\left(\frac{1}{\sqrt{6}}|000\rangle\underbrace{\langle 000|000\rangle}_{1} + \cancel{\frac{1}{\sqrt{6}}|111\rangle\underbrace{\langle 111|000\rangle}_{0}} + \dots\right)$$

$$= \left(\frac{1}{\sqrt{6}}\langle 000| + \sqrt{\frac{2}{6}}\langle 001| + \sqrt{\frac{3}{6}}\langle 110|\right)\left(\frac{1}{\sqrt{6}}|000\rangle\right) = \frac{1}{6}$$

Prob of P

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle$ with M and get outcome $P_1 = |001\rangle\langle 001| + |110\rangle\langle 110|$ what is the resultant state?

$$|\psi\rangle \to \frac{P_i|\psi\rangle}{\sqrt{Pr(i)}} \qquad Pr(i) = \langle\psi|P_i|\psi\rangle$$

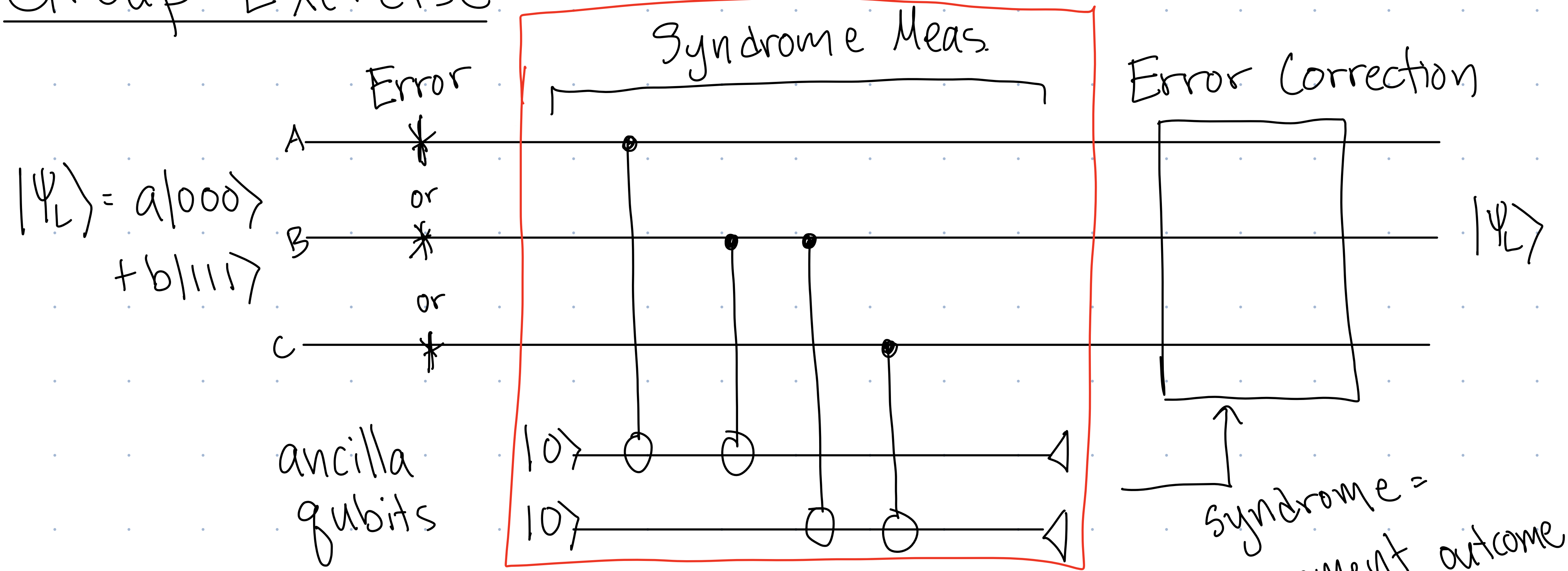
$$P_i|\psi\rangle = \left(|001\rangle\langle 001| + |110\rangle\langle 110|\right)\left(\frac{1}{\sqrt{6}}|00\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)$$

$$Pr(P_1) = \frac{5}{6}$$

$$\sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\underbrace{\langle 110|110\rangle}_{1}$$

$$\to \frac{\sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle}{\sqrt{\frac{5}{6}}} = \sqrt{\frac{2}{5}}|001\rangle + \sqrt{\frac{3}{5}}|110\rangle$$

Group Exercise



syndrome = measurement outcome of ancilla qubits

$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,\ |010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$

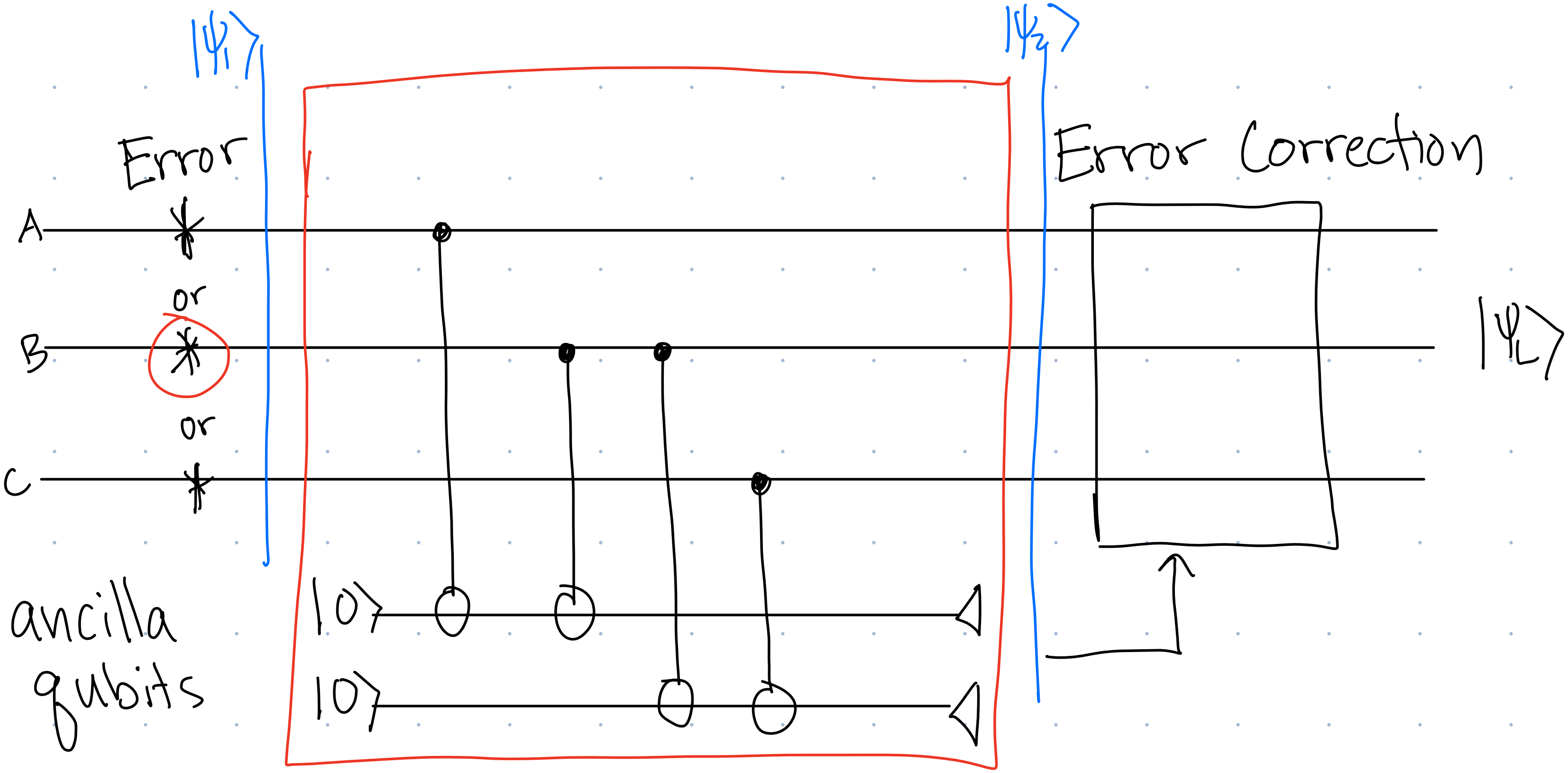
$*$: $R_x(\theta) = \cos\theta\, I + i \sin\theta\, X$

ex: $R_x(\theta)|1\rangle = \cos\theta\, I|1\rangle + i\sin\theta\, X|1\rangle = \cos\theta|1\rangle + i\sin\theta|0\rangle$

1. What are possible outcomes + what does state collapse to?
2. What error correction procedure should be used in each case?

Solution

$|\psi_L\rangle = a|000\rangle + b|111\rangle$



$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,$
$|010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$

$a|000\rangle \rightarrow a\cos\theta|000\rangle + ai\sin\theta|010\rangle$

$|\psi_1\rangle = a\cos\theta|000\rangle + ai\sin\theta|010\rangle + b\cos\theta|111\rangle + bi\sin\theta|101\rangle$

$=$

$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,$
$|010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$

$a|000\rangle \to a\cos\theta|000\rangle + ai\sin\theta|010\rangle$

$|\psi_1\rangle = a\cos\theta|000\rangle + ai\sin\theta|010\rangle + b\cos\theta|111\rangle + bi\sin\theta|101\rangle$

$P_0:$ $\boxed{P_0|\psi_1\rangle = a\cos\theta|000\rangle + b\cos\theta|111\rangle}$

$\langle\psi_1|P_0|\psi_1\rangle = (a^*\cos\theta\langle 000| - a^* i\sin\theta\langle 010| + b^*\cos\theta\langle 111| - b^* i\sin\theta\langle 101|)$
$(a\cos\theta|000\rangle + b\cos\theta|111\rangle)$
$= a^*a\cos^2\theta + b^*b\cos^2\theta = \cos^2\theta(\underbrace{|a|^2 + |b|^2}_{1})$
$= \boxed{\cos^2\theta = \text{Prob of } P_0}$

Collapsed State:
$$\frac{a\cos\theta|000\rangle + b\cos\theta|111\rangle}{\sqrt{\cos^2\theta}} = a|000\rangle + b|111\rangle$$

$P_2:$ $P_2|\psi_1\rangle = ai\sin\theta|010\rangle + bi\sin\theta|101\rangle$ Prob: $\sin^2\theta$

Collapsed State $\dfrac{P_2|\psi_1\rangle}{\sqrt{\sin^2\theta}} = ai|010\rangle + bi|101\rangle$

Apply $-iX_3 \Rightarrow a|000\rangle + b|111\rangle$

Error on A qubit: P_0 or $P_3 \Rightarrow X_A$

Error on B qubit: P_0 or $P_2 \Rightarrow X_B$

Error on C qubit: P_0 or $P_1 \Rightarrow \underbrace{X_C \qquad}_{\text{Correct}}$ $\Big\}\ a|000\rangle + b|111\rangle$

$\Downarrow$

Collapse $a|000\rangle + b|111\rangle$

Big Idea with ~~Quantum~~ Error Correction

- Partial collapse turns continuous error $R_x(\theta)$ into discrete error (X, I)
- Partial meas. provides info on how to correct while preserving superposition

Need to Correct More Errors? Bigger Code!

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

9-qubit Shor Code

Corrects any single qubit error

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9-qubit Shor Code

Corrects any single qubit error

Z_6 error:

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

Z_1 and Z_4

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

Which errors can be corrected by Shor's code?

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

A) $Z_1 Z_2$

B) $Z_4 Z_8$

C) $Z_1 Z_2 Z_3$

D) $X_1 Z_2$

E) $X_1 X_2$

F) $X_1 X_4$

G) $Y_1 Z_4$

Fault Tolerant Quantum Computing

Shor Code: All single qubit errors, 9 qubits, 1 logical qubit

Concatenation:

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

But... more qubits means more chance of error. Can only keep ahead of accumulating errors if error rate is small enough