Error Correction

Learning Goals

- Describe errors in quantum/classical systems
- Describe repetition codes + why they work
- Analyze how a repetition code corrects errors

Announcements

- No Tues OH
- Come to OH
- Project Q's?

Exit Tickets

- Error During E.C.?
- Projective Meas → shortcut OK

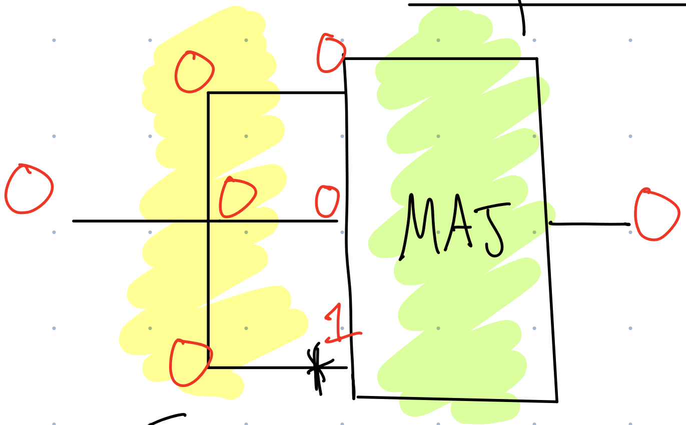
If Q. Computers are so great, why haven't we built one?

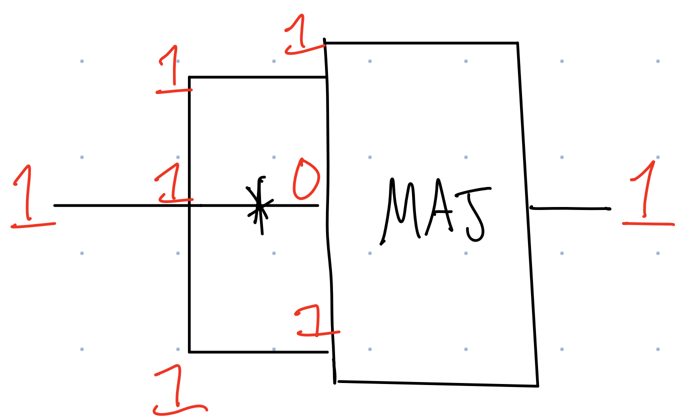
ERRORS - unwanted gates + measurement

- Classical Computers
  - Cosmic Rays → 1 bit flip / 4 GB / per day

- Super mario glitch
- Pace maker
- Plane dropped

Solution: Repetition Code





Fanout

"Encoding" "Error Correction"

- Sources of Quantum Errors
  - control laser imperfect
  - imperfect vacuum
  - stray electric + magnetic fields
  - anomalous heating

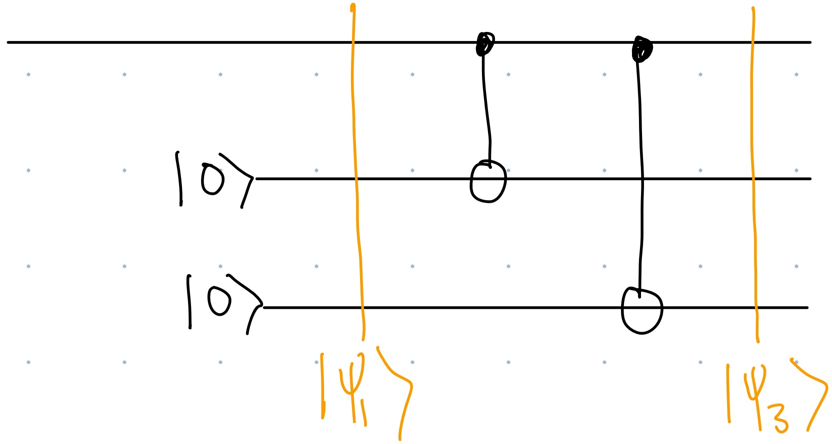
Why won't repetition code work for quantum?

A) MAJ not reversible
B) Doesn't correct phase errors (Z)
C) No fanout because can't copy quant
D) There are uncountably infinitely many errors to deal with

Shor's Repetition Code

Encoding:

$|\psi\rangle = a|0\rangle + b|1\rangle$



original state to protect from errors. Don't know a, b.

$|\psi_1\rangle = (a|0\rangle + b|1\rangle)|0\rangle|0\rangle$

$|\psi_2\rangle =$

$=$

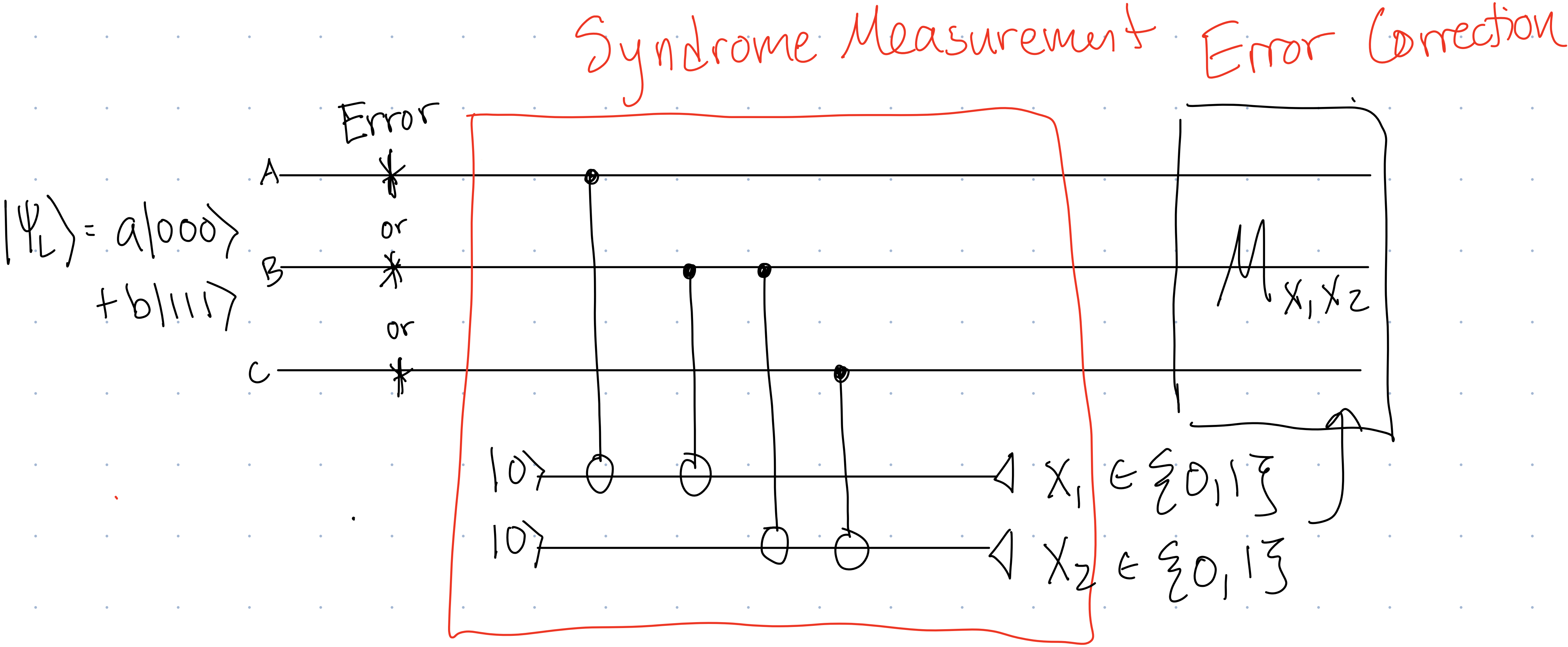
physical qubits

$|\psi_3\rangle = a\underbrace{|000\rangle}_{|0\rangle} + b\underbrace{|111\rangle}_{|1\rangle}$ ← logical qubit

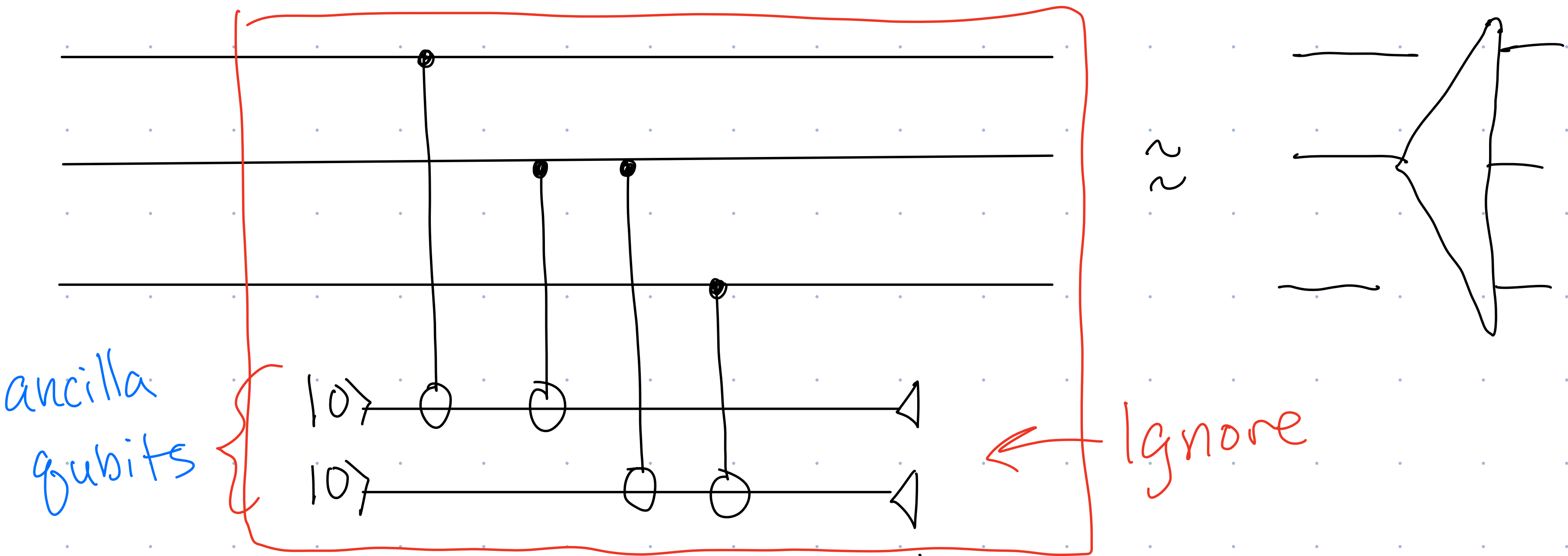
logical ket 0

logical ket 1

Error Correction Circuit



Syndrome Meas.



Not another partial measurement!

New approach: Projective Measurement

Projective Measurement

Described by a
Set of projectors

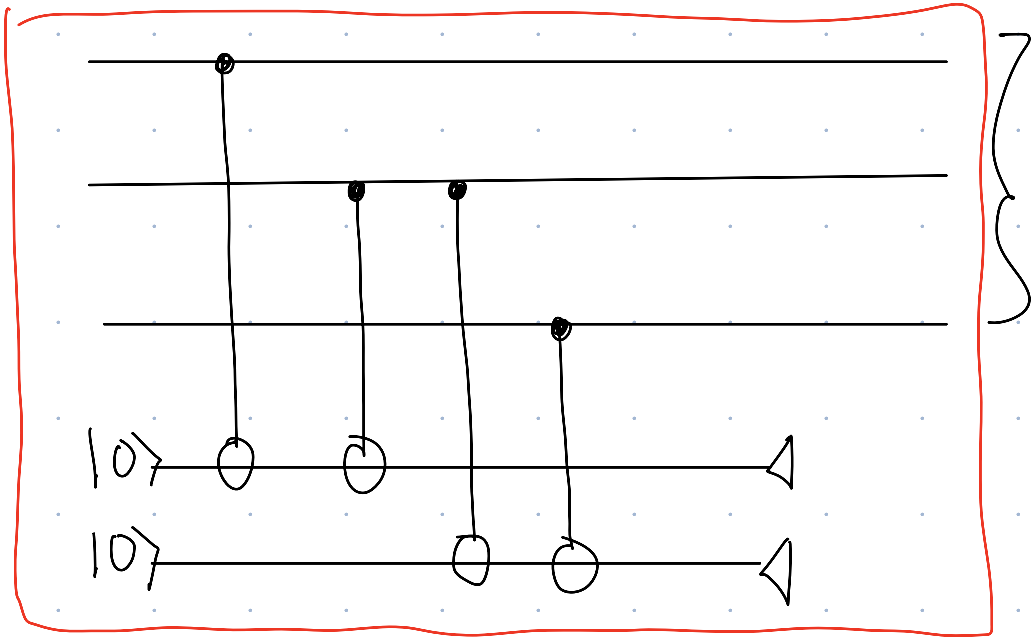
$$P_0 = |\phi_1\rangle\langle\phi_1| + |\phi_4\rangle\langle\phi_4|$$
$$P_1 = |\phi_3\rangle\langle\phi_3|$$
$$P_2 = |\phi_2\rangle\langle\phi_2| + |\phi_5\rangle\langle\phi_5| + |\phi_6\rangle\langle\phi_6|$$
$$\vdots$$

$$\boxed{\text{techical: } \sum_i P_i = I}$$

Measure $|\psi\rangle$

Probability of outcome P_i : $\langle\psi|P_i|\psi\rangle$

After outcome i, state collapses to $\dfrac{P_i|\psi\rangle}{\sqrt{Pr(i)}}$



$$M = \{ \underbrace{|000\rangle\langle 000| + |111\rangle\langle 111|}_{P_0},\ \underbrace{|001\rangle\langle 001| + |110\rangle\langle 110|}_{P_1},\ \underbrace{|010\rangle\langle 010| + |101\rangle\langle 101|}_{P_2},\ \underbrace{|100\rangle\langle 100| + |011\rangle\langle 011|}_{P_3} \}$$

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle$ with M, which outcomes are possible?

$$\boxed{|\psi\rangle \to \frac{P_i|\psi\rangle}{\sqrt{Pr(i)}} \qquad Pr(i) = \langle\psi|P_i|\psi\rangle}$$

A) P_0 $\quad Pr(P_0) = \frac{1}{6}$

B) P_1 $\quad Pr(P_1) = \frac{5}{6}$

C) P_2

D) P_3

P_0

$$\langle\psi| \left(|000\rangle\langle 000| + |111\rangle\langle 111|\right) \left(\frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)$$

$$\frac{1}{\sqrt{6}}|000\rangle\underbrace{\langle 000|000\rangle}_{1} + \sqrt{\frac{2}{6}}|000\rangle\langle 000|001\rangle + \dots$$

$$\langle\psi|\frac{1}{\sqrt{6}}|000\rangle \rightarrow \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}\langle 000|000\rangle = \frac{1}{6}$$

If measure $|\psi\rangle = \frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle$ with M and get outcome $P_1 = |001\rangle\langle 001| + |110\rangle\langle 110|$ what is the resultant state?

$$|\psi\rangle \to \frac{P_i|\psi\rangle}{\sqrt{Pr(i)}} \qquad Pr(i) = \langle\psi|P_i|\psi\rangle$$

$$P_1|\psi\rangle = \Big(|001\rangle\langle 001| + |110\rangle\langle 110|\Big)\left(\frac{1}{\sqrt{6}}|000\rangle + \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)$$

$$= \sqrt{\frac{2}{6}}|001\rangle\left(\cancelto{1}{\langle 001||001\rangle}\right) + \sqrt{\frac{3}{6}}|110\rangle\left(\cancelto{1}{\langle 110||110\rangle}\right)$$

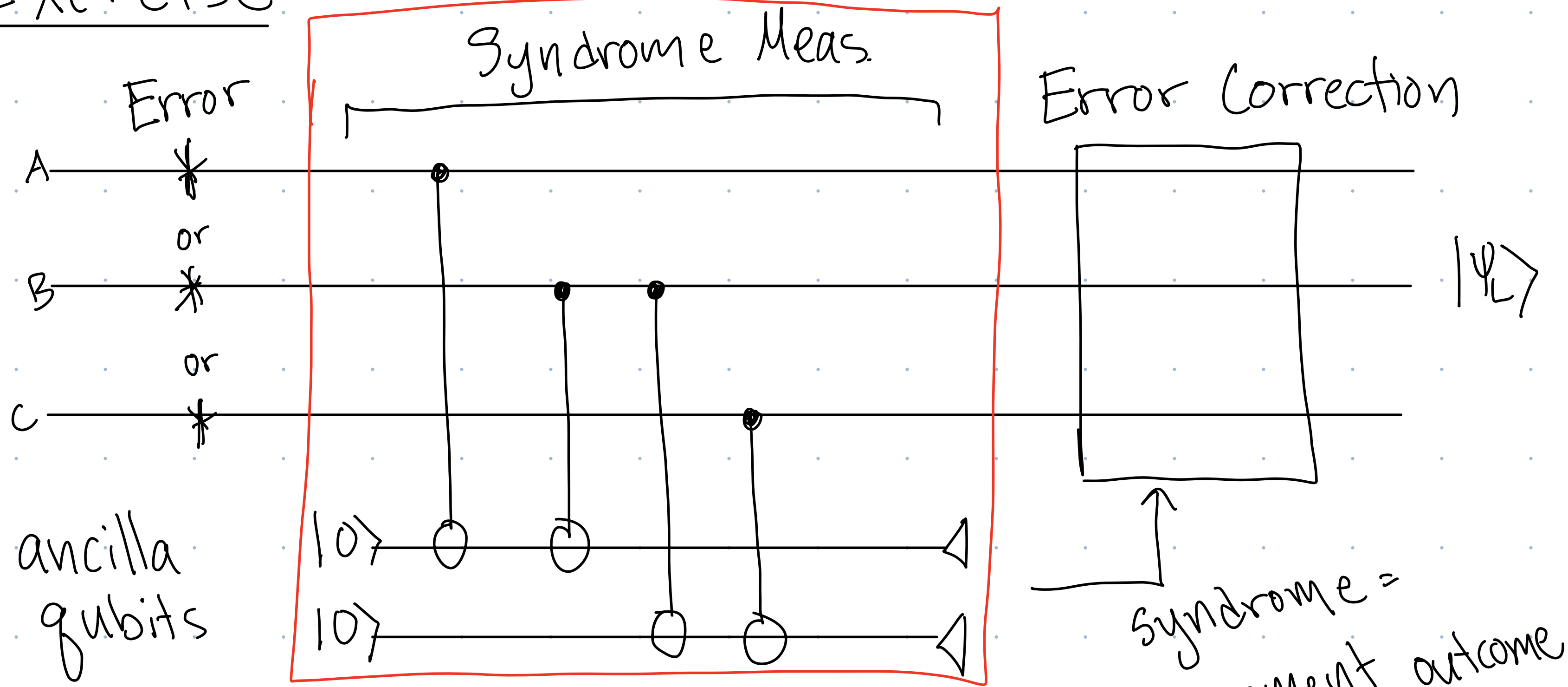
$$P_1|\psi\rangle = \left(\sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle\right)$$

$$\frac{1}{\sqrt{Pr}}P_1|\psi\rangle = \left(\sqrt{\frac{2}{\cancel{6}}}|001\rangle + \sqrt{\frac{3}{\cancel{6}}}|110\rangle\right)\sqrt{\frac{\cancel{6}}{5}}$$

$$= \sqrt{\frac{2}{5}}|001\rangle + \sqrt{\frac{3}{5}}|110\rangle$$

Group Exercise

$|\psi_L\rangle = a|000\rangle + b|111\rangle$



$$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,\ |010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$$

$$\boxed{|\psi\rangle \rightarrow \frac{P_i|\psi\rangle}{\sqrt{P_r(i)}} \qquad P_r(i) = \langle\psi|P_i|\psi\rangle}$$

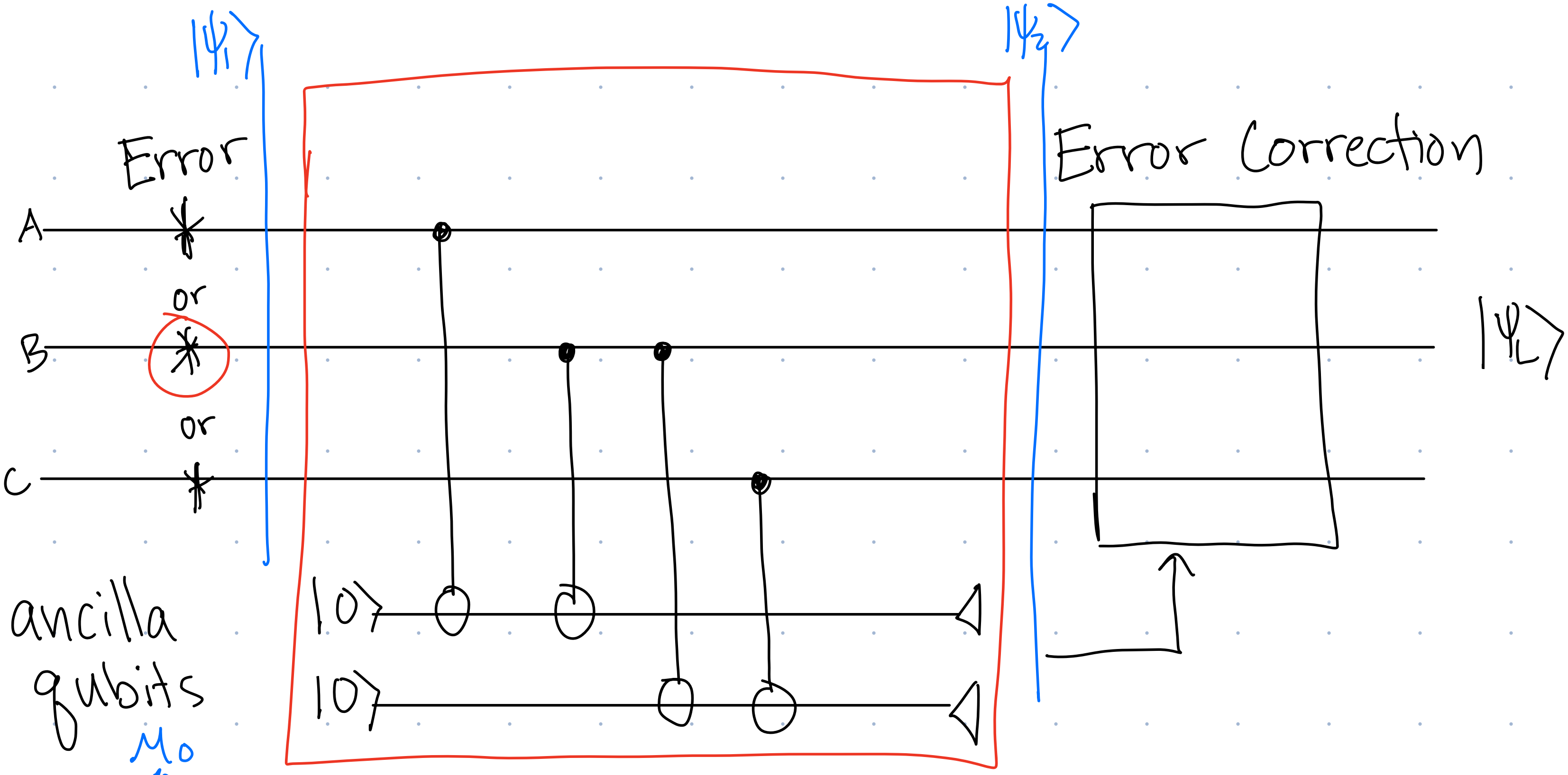
$*$: $R_x(\theta) = \cos\theta\, I + i\sin\theta\, X$

ex: $R_x(\theta)|1\rangle = \cos\theta\, I|1\rangle + i\sin\theta\, X|1\rangle = \cos\theta|1\rangle + i\sin\theta|0\rangle$

1. What are possible outcomes + what does state collapse to?
2. What error correction procedure should be used in each case?

Solution

$$|\psi_L\rangle = a|000\rangle + b|111\rangle$$



M_0

$$M = \{ |000\rangle\langle 000| + |111\rangle\langle 111|,\ |001\rangle\langle 001| + |110\rangle\langle 110|,$$

$$M_2 \leftarrow |010\rangle\langle 010| + |101\rangle\langle 101|,\ |100\rangle\langle 100| + |011\rangle\langle 011| \}$$

$$R_x(\theta)_B \left(a|000\rangle + b|111\rangle\right) = \left(\cos\theta I + i\sin\theta X\right)\left(a|000\rangle + b|111\rangle\right)$$

$$= a\cos\theta|000\rangle + b\cos\theta|111\rangle + ia\sin\theta|010\rangle + ib\sin\theta|101\rangle$$

M_0

$$M_0|\psi\rangle = a\cos\theta|000\rangle + b\cos\theta|111\rangle$$

$$\langle\psi|M_0|\psi\rangle = \left(a^*\cos\theta\langle 000| + b^*\cos\theta\langle 111|\right)\left(a\cos\theta|000\rangle + b\cos\theta|111\rangle\right)$$

$$= |a|^2\cos^2\theta + |b|^2\cos^2\theta = \cos^2\theta\left(|a|^2 + |b|^2\right)$$

$Pr(M_0) = \cos^2\theta$

Original state: $a|0\rangle + b|1\rangle$

Normalization: $|a|^2 + |b|^2 = 1$

Collapse: $\dfrac{M_0|\psi\rangle}{\sqrt{Pr(M_0)}} = \dfrac{a\cos\theta|000\rangle + b\cos\theta|111\rangle}{\sqrt{\cos^2\theta}}$

$= a|000\rangle + b|111\rangle$

M_2

$M_2|\psi\rangle = \boxed{ia\sin\theta|010\rangle + ib\sin\theta|101\rangle}$

$\langle\psi|M_2|\psi\rangle = \left(-ia^*\sin\theta\langle 010| - ib^*\sin\theta\langle 101|\right)$

$= |a|^2\sin^2\theta + |b|^2\sin^2\theta$

$Pr(M_2) = \sin^2\theta$

global phase

Collapse $\dfrac{M_2|\psi\rangle}{\sqrt{Pr(M_2)}} = \dfrac{\cancel{i}a\sin\theta|010\rangle + \cancel{i}b\sin\theta|101\rangle}{\sqrt{\sin^2\theta}}$

$= a|010\rangle + b|101\rangle$

Recover: $I \otimes X \otimes I \Rightarrow a|000\rangle + b|111\rangle$

Error on A qubit: Outcome or

Error on B qubit Outcome or

Error on C qubit Outcome or

Big Idea with Error Correction

- Partial collapse turn $\overbrace{\text{continuous errror}}^{R_x(\theta)}$ into a $\underbrace{\text{discrete error}}_{I,\ X}$
- Partial collapse give info on how to correct while preserving a/b superposition

Rules for Correcting Proj. Meas

Outcome

$$P_1 = \{ |\psi_a\rangle\langle\psi_a| + |\psi_b\rangle\langle\psi_b| + \dots \} \simeq \{ |\psi_a\rangle, |\psi_b\rangle, \dots \}$$

If get outcome P_1, state collapses to

$$\alpha |\psi_a\rangle + \beta |\psi_b\rangle \quad (\alpha, \beta \text{ unknown})$$

We want to be in

$$\gamma |0_L\rangle + \eta |1_L\rangle \quad (\gamma, \eta \text{ unknown})$$

Correct by finding lowest weight correction

Correction Rule:

* Find lowest weight correction

$$\alpha|000\rangle + \beta|111\rangle \Updownarrow \alpha|0\rangle + \beta|1\rangle$$

M_1

ex: Outcome $\{|100\rangle, |011\rangle\} \xrightarrow{\text{collapse}} |\psi\rangle \to \alpha|100\rangle + \beta|011\rangle$

- If apply X_1 $\Rightarrow \alpha|000\rangle + \beta|111\rangle$

  weight $\cdot$ 1

  In codespace (in a superposition of $|000\rangle$, $|111\rangle$)

- If apply X_2X_3 $\Rightarrow \alpha|111\rangle + \beta|000\rangle \Leftrightarrow \alpha|1\rangle + \beta|0\rangle$

  weight 2

  Back in codespace

  Logical X

Need to Correct More Errors? Bigger Code!

$$a|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$b|1_L\rangle = \left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

} 9-qubit Shor Code

Corrects any single qubit error

Z_6 error:

$$Z_6|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} \overset{-}{\cancel{+}} |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$Z_6|1_L\rangle = \left(|000\rangle - |111\rangle\right)\left(|000\rangle \overset{+}{\cancel{-}} |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

Z_1 and Z_4

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

Need to Correct More Errors? Bigger Code!

$$\left.\begin{aligned}
|0_L\rangle &= \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right) \\
|1_L\rangle &= \left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)
\end{aligned}\right\} \text{9-qubit Shor Code}$$

Corrects any single qubit error

Easiest way to return to code space?

Which errors can be corrected by Shor's code?

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle_{123} - |111\rangle_{123}\right)\left(|000\rangle_{456} - |111\rangle_{456}\right)\left(|000\rangle_{789} - |111\rangle_{789}\right)$$

A) $Z_1 Z_2$

B) $Z_4 Z_8$

C) $Z_1 Z_2 Z_3$

D) $X_1 Z_2$

E) $X_1 X_2$

F) $X_1 X_4$

G) $Y_1 Z_4$

Fault Tolerant Quantum Computing

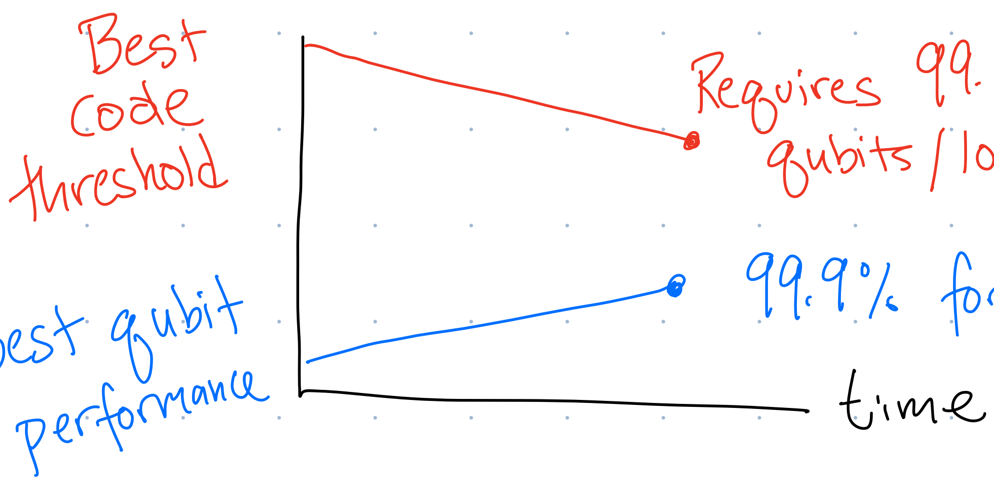
Shor Code: All single qubit errors, 9 qubits, 1 logical qubit
Concatenation: All 2-qubit errors, 81 qubits, 1 logical qubit

$$|0_L\rangle = \left(|000\rangle_{123} + |111\rangle_{123}\right)\left(|000\rangle_{456} + |111\rangle_{456}\right)\left(|000\rangle_{789} + |111\rangle_{789}\right)$$

$$|1_L\rangle = \left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

$$|0_L\rangle$$

But... more qubits means more chance of error. Can only keep ahead of accumulating errors if error rate is small enough



Requires 99.9% accuracy, 1000–10,000 physical qubits / logical qubit

99.9% for 2 qubits, 99.99% for 1 qubit
"fluxonium qubit"

E