Learning Goals

- Design + analyze a quant. algorithm
- Read circuit diagrams
- Describe time + query complexity

cheatsheet
3"x5" notecard

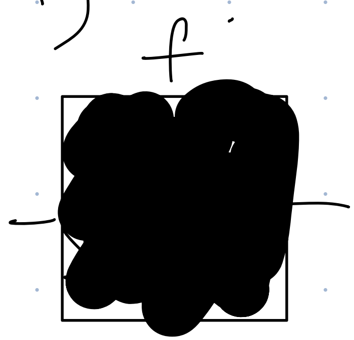
Focus

Announcements

- In-class long exam (2 weeks)
  - Q12, Q13, QC1 (All foundational)
  - Q14, QC2 (interm.)
  - Q15 (advanced)

Exit Tickets

- If f is not a "black box",



how many queries?
- More than 2 queries useful?
- Why I?

U_A Nothing B $\qquad U_A \otimes I_B$

- Gates/Unitaries to left of state/ket:

$$U|\psi\rangle = |\psi'\rangle$$

- Multiple gates:

$$U_3 U_2 U_1 |\psi\rangle$$

1st applied

2nd applied

What is query complexity?

What is time complexity?

Deutsch's algorithm analyzes a 1-bit function f:

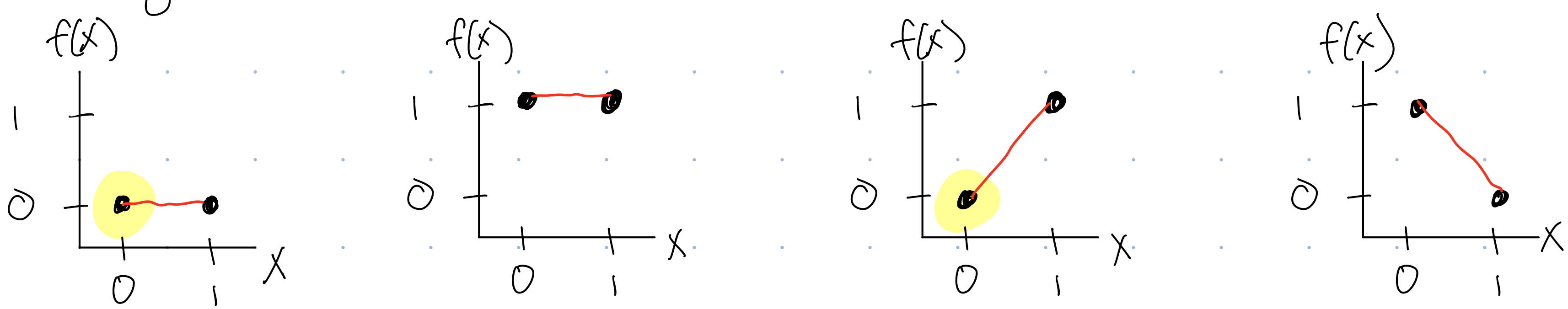
x	$f(x)$
0	$f(0)$
1	$f(1)$

$$f(0), f(1) \in \{0, 1\}$$

ex: $f(x) = \begin{cases} 1 & \text{if it will rain at time } x \\ 0 & \text{if it will not rain at time } x \end{cases}$

$x = \begin{cases} 0 \to \text{day} \\ 1 \to \text{night} \end{cases}$

Only 4 1-bit Functions:

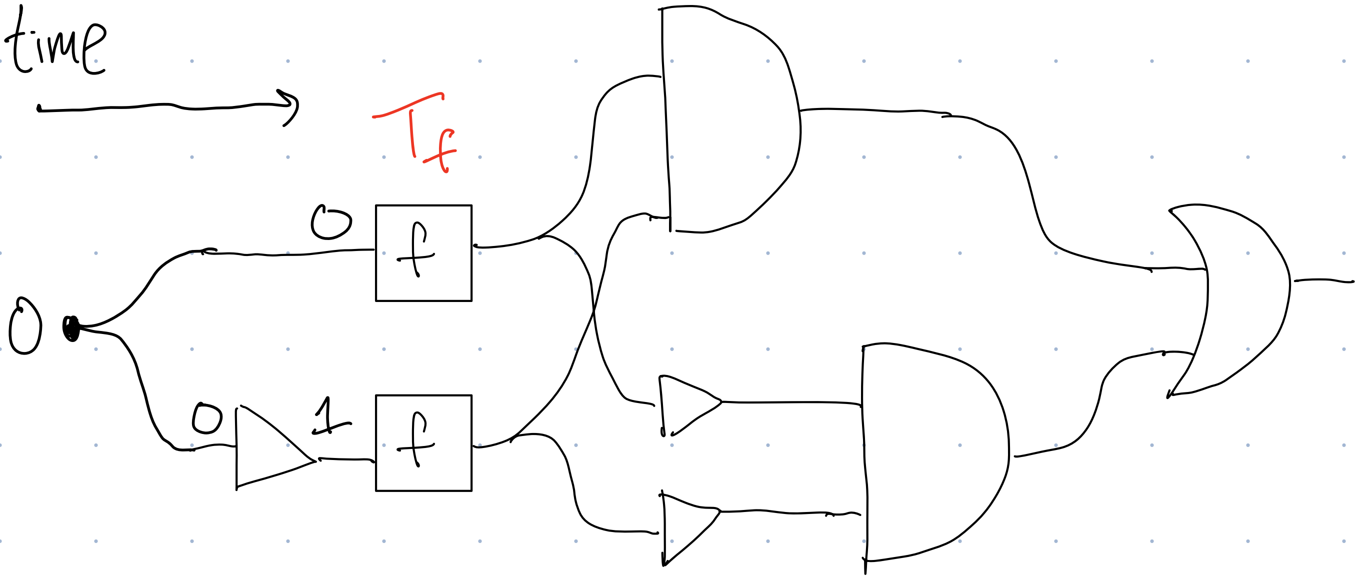


flat (even) sloped (balanced)

Deutsch's Problem: Given query access to f, determine if flat or sloped

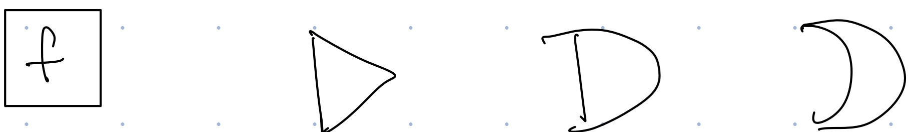
A Classical Algorithm (circuit diagram)

time



- time goes left → right
- wires are bits
- shapes are gates

1 ⇒ flat
0 ⇒ sloped



"query to f" NOT AND OR

# of times the gate f is used

Query Complexity = 2

Time Complexity = time to run

$2T_f + 6$ // $T_f + 3$

We would like to determine if f is flat or sloped using as few queries as possible.

What is the minimum number of classical queries to f needed to determine flat/sloped

A) 0

B) 1

C) 2

Deutsch's Problem

- Classical Query Complexity: 2
- Quantum Query Complexity: $\underline{1}$

Why do we care?!

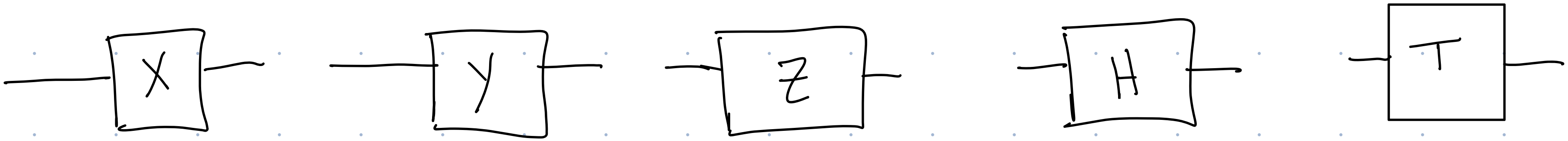
Quantum Gates (Circuit picture)

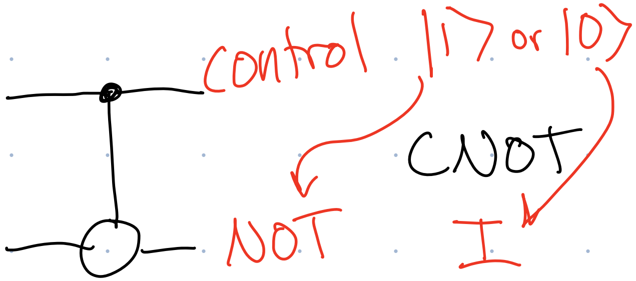
- time L → R
- wires are qubits
- Gates + measurements are shapes

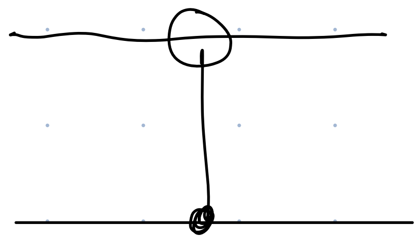
Time →

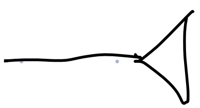
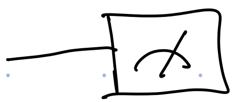
$|0\rangle$ 

Initialize each qubit $|0\rangle$ (usually)







 or 

measure in standard basis

$M = \{|0\rangle, |1\rangle\}$

Need gate for f. What about:

U_f

$\cancel{|0\rangle \to |f(0)\rangle}$

$\cancel{|1\rangle \to |f(1)\rangle}$

$f(0) = f(1) = 1$ (flat)

$|0\rangle \to |1\rangle$

$|1\rangle \to |1\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \to$

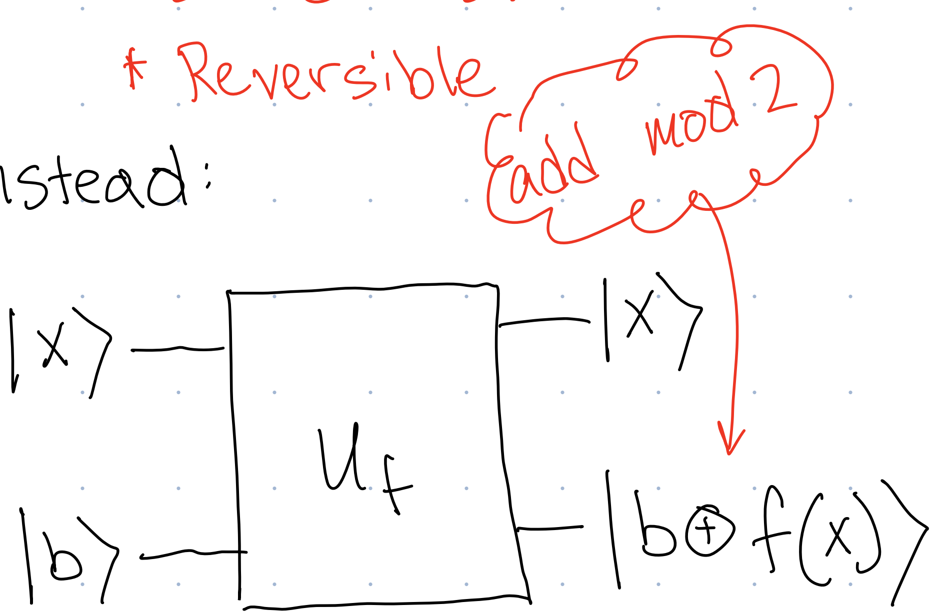
$\frac{1}{\sqrt{2}}(|1\rangle + |1\rangle) = \frac{2}{\sqrt{2}}|1\rangle$

Explain why this is not an allowed gate.

* State $\to$ State
* Reversible

$U_f U_f = I$

Instead:



$\downarrow U_f$

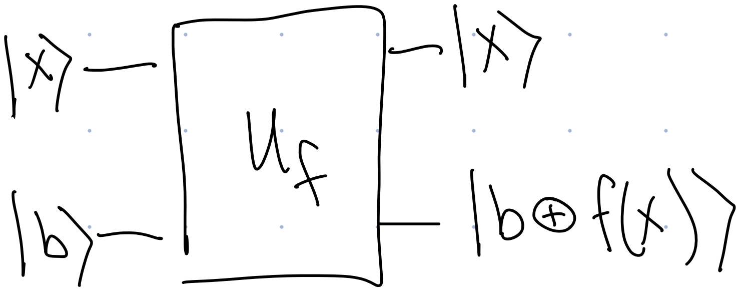
$|b \oplus \cancel{f(x) \oplus f(x)}\rangle$ 0

$x, b \in \{0, 1\}$

U_f

$\boxed{|x\rangle|b\rangle \to |x\rangle|b \oplus f(x)\rangle}$

This only works if
x, b are bits
$|x\rangle, |b\rangle$ are standard basis states

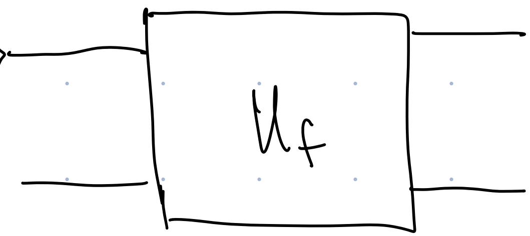


If $x=0$ get info about $f(0)$

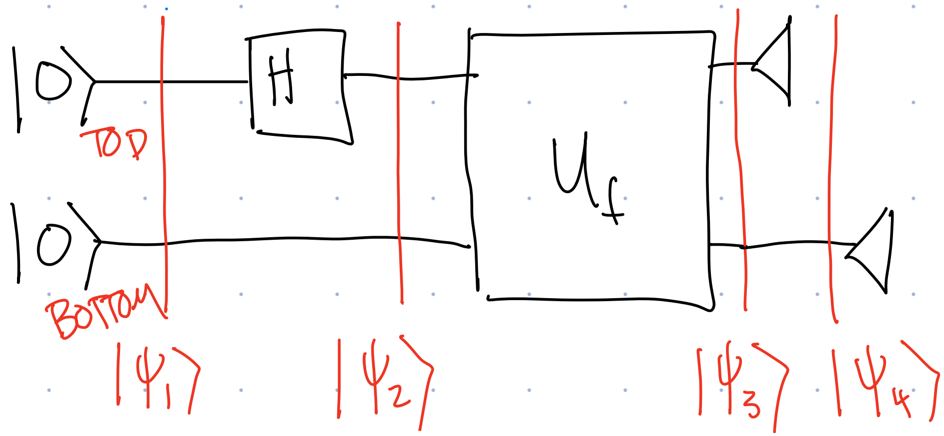
If $x=1$ get info about $f(1)$

Idea – Superposition!

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



Analyzing a Quantum Circuit



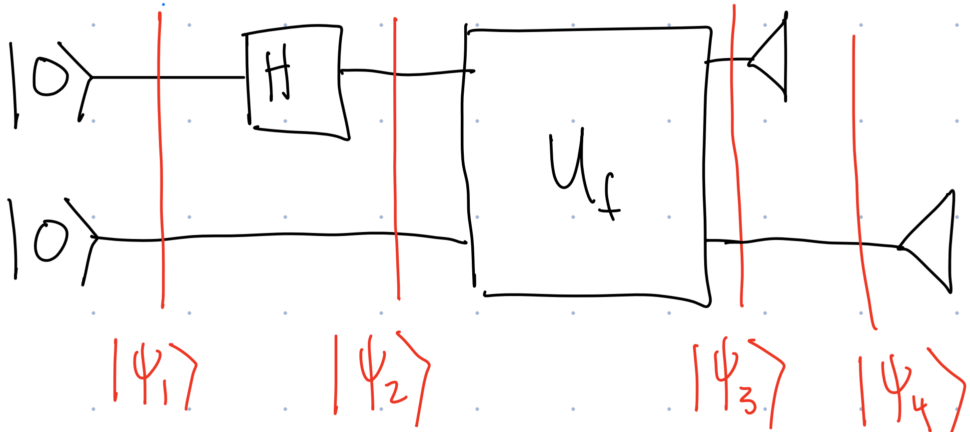
$$\boxed{|x\rangle|b\rangle \to |x\rangle|b\oplus f(x)\rangle}$$

$$|0\rangle|0\rangle \to |0\rangle|0\oplus f(0)\rangle$$
$$= |0\rangle|f(0)\rangle$$
$$|1\rangle|0\rangle \to |1\rangle|0\oplus f(1)\rangle$$
$$|1\rangle|f(1)\rangle$$

$$|\psi_1\rangle = \underbrace{|0\rangle}_{\text{TOP}}\underbrace{|0\rangle}_{\text{BOTTOM}}$$

$$|\psi_2\rangle = (H\otimes I)|0\rangle|0\rangle = |+\rangle|0\rangle$$

$$|\psi_3\rangle = U_f\left(|+\rangle|0\rangle\right) = \cancel{|+\rangle|0\oplus f(+)\rangle}$$
$$= U_f\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle = U_f\left(\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle\right)$$
$$= \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle$$
$$=$$



$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle$$
$$= |0\rangle\left(\tfrac{1}{\sqrt{2}}|f(0)\rangle\right) + |1\rangle\left(\tfrac{1}{\sqrt{2}}|f(1)\rangle\right)$$

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Partial Measurement:

Outcome $|0\rangle$

- Prob $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- Collapse $\frac{1}{\frac{1}{\sqrt{2}}}\left(\frac{1}{\sqrt{2}}|f(0)\rangle\right) = |f(0)\rangle$

Outcome $|1\rangle$

- Prob $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- Collapse $\longrightarrow$ $|f(1)\rangle$

Mini-Hype Lesson

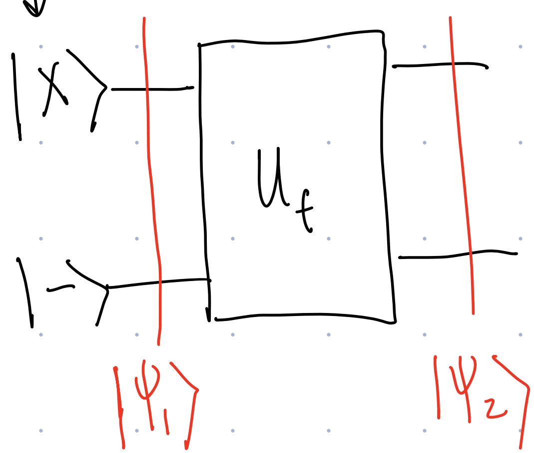
You need more than superposition to get a quantum advantage.

$$U_f$$

$$\boxed{|x\rangle|b\rangle \to |x\rangle|b \oplus f(x)\rangle}$$

Group Exercise:

$x = 0$ or 1

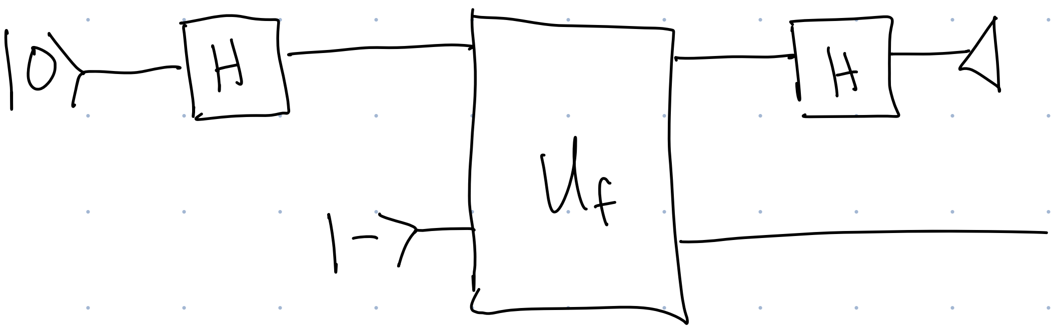


Show that $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|-\rangle$

If $f(x) = 0$

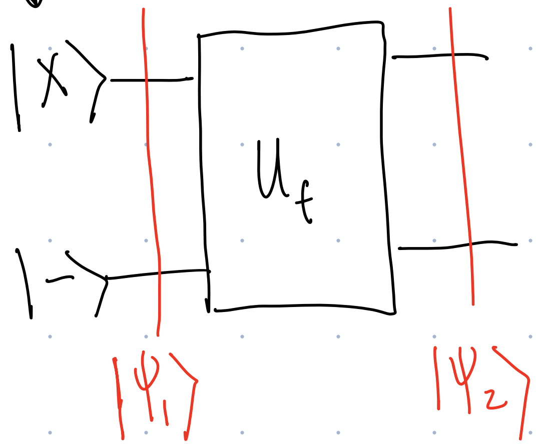
If $f(x) = 1$

Analyze outcome of this circuit:



Solution

$x = 0$ or 1



$$\boxed{|x\rangle|b\rangle \to |x\rangle|b \oplus f(x)\rangle}$$

Show that $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|-\rangle$

$$|\psi_2\rangle = U_f|x\rangle|-\rangle = U_f|x\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$
$$= U_f\left(\tfrac{1}{\sqrt{2}}|x\rangle|0\rangle - \tfrac{1}{\sqrt{2}}|x\rangle|1\rangle\right)$$
$$= \tfrac{1}{\sqrt{2}}U_f|x\rangle|0\rangle - \tfrac{1}{\sqrt{2}}U_f|x\rangle|1\rangle$$
$$= \tfrac{1}{\sqrt{2}}|x\rangle|0 \oplus f(x)\rangle - \tfrac{1}{\sqrt{2}}|x\rangle|1 \oplus f(x)\rangle$$

$(-1)^0 = 1$ $\quad (-1)^1 = -1$

New rule for U_f:

$$|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$$

$|x\rangle$: standard basis

$f(x) = 0$

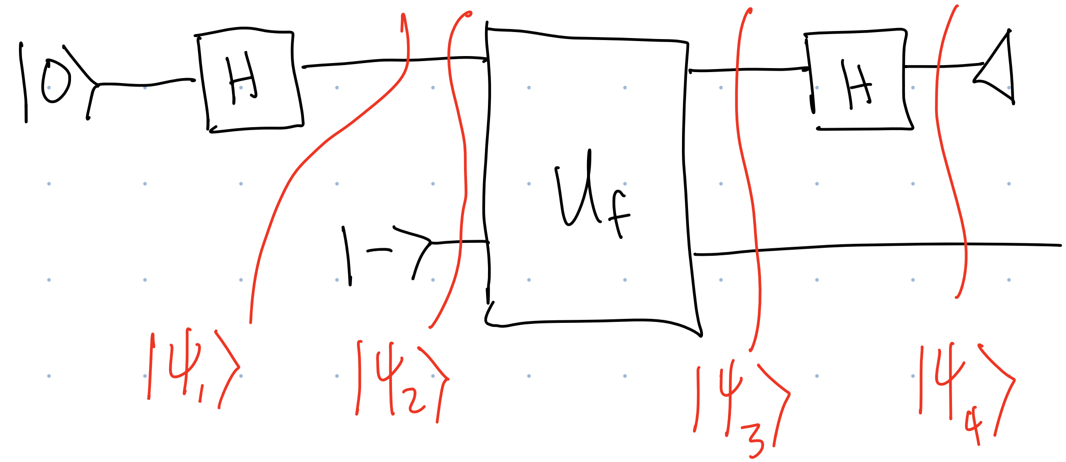
$$\tfrac{1}{\sqrt{2}}|x\rangle|0\rangle - \tfrac{1}{\sqrt{2}}|x\rangle|1\rangle$$
$$|x\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$
$$= |x\rangle|-\rangle$$

$f(x) = 1$

$$\tfrac{1}{\sqrt{2}}|x\rangle|1\rangle - \tfrac{1}{\sqrt{2}}|x\rangle|0\rangle$$
$$|x\rangle\left(-\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$
$$-|x\rangle|-\rangle$$

Phase Kickback

$$U_f : |x\rangle|-\rangle \to (-1)^{f(x)}|x\rangle|-\rangle$$



$|\psi_1\rangle =$

$|\psi_2\rangle =$

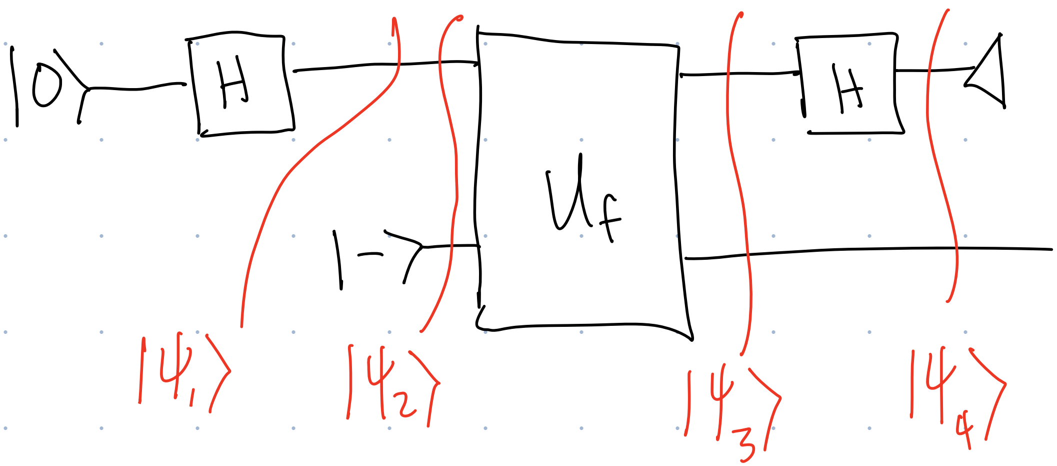
$|\psi_3\rangle =$

$=$

$=$

$=$

Deutsch's Alg:



Phase Kickback