Deutsch's Algorithm

Learning Goals

- Design + analyze a quant. algorithm
- Read circuit diagrams
- Describe time + query complexity

Announcements

- What I Did This Summer go/WIDTS 11/4, 11/11 12:30-1:30
- go/SK_F24_checkin
- grade shift
- No OH Friday

Exit Tickets

Total measurement

Query needs to be quantum

More qubits?

Deutsch's algorithm analyzes a 1-bit function f:

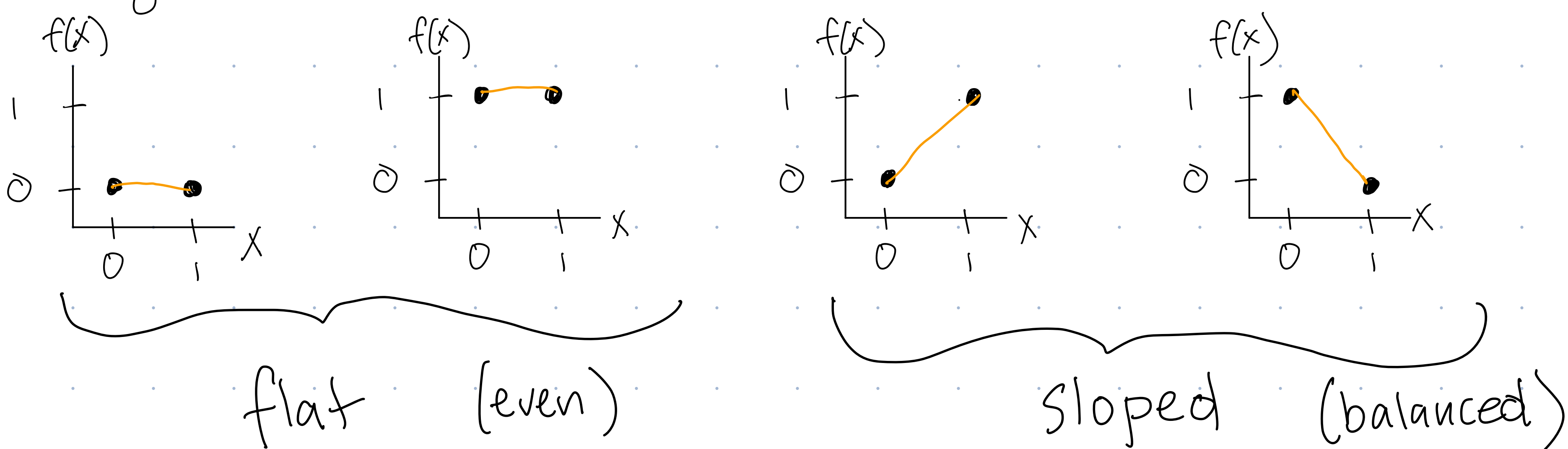
x	$f(x)$
0	$f(0)$
1	$f(1)$

$f(0), f(1) \in \{0,1\}$

ex: $f(x) = \begin{cases} 1 & \text{if it will rain at time } x \\ 0 & \text{if it will not rain} \end{cases}$

$x = \begin{cases} 0 \rightarrow \text{day} \\ 1 \rightarrow \text{night} \end{cases}$

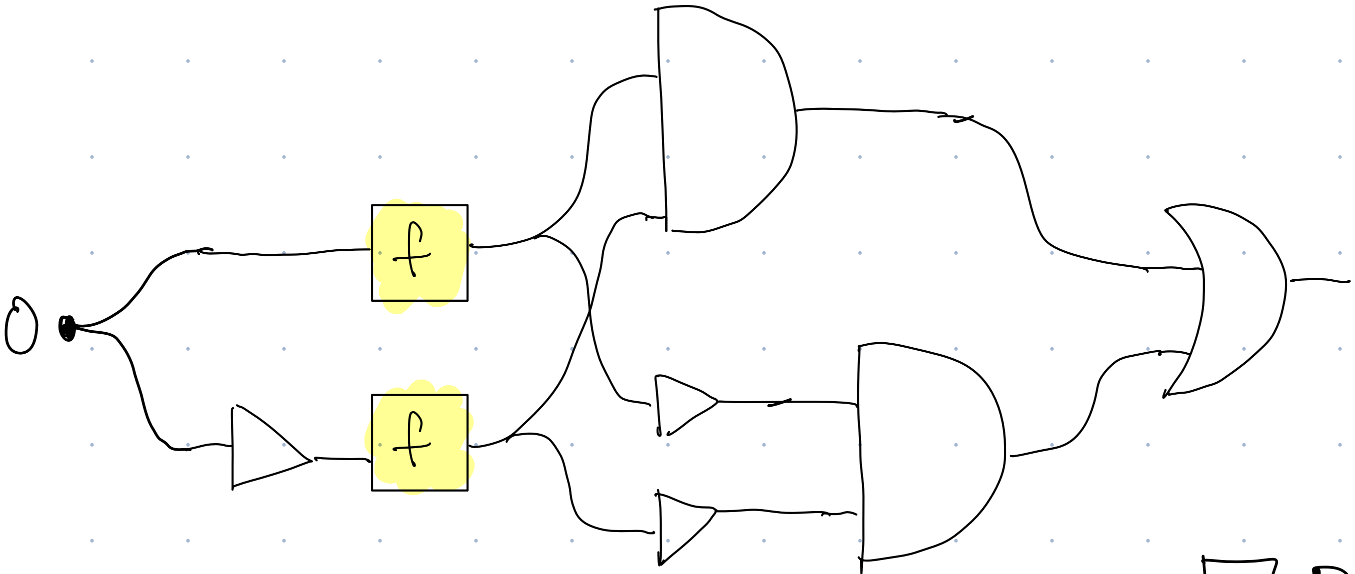
Only 4 1-bit Functions:



Deutsch's Problem: Given query access to f, determine if f is flat or sloped

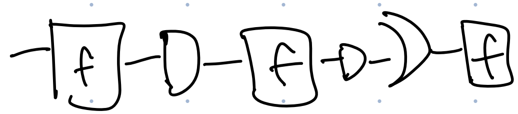
A Classical Algorithm (circuit diagram)

- time goes L → R
- Wires are bits
- Shapes are gates



1 ↔ flat

0 ↔ sloped



Query Complexity = 2

Time Complexity = 5, 8, time complexity of f?

What is the minimum number of classical queries to f needed to determine flat/sloped

A) 0 B) 1 C) 2

Deutsch's Problem

- Classical Query Complexity: 2
- Quantum Query Complexity: 1

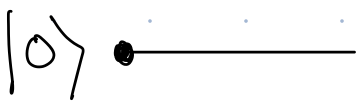
Why do we care?!

- Provable quantum computing advantage
- Techniques

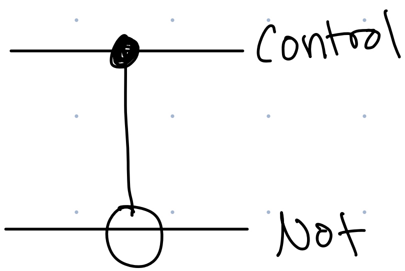
- Might be useful if f is really hard to compute

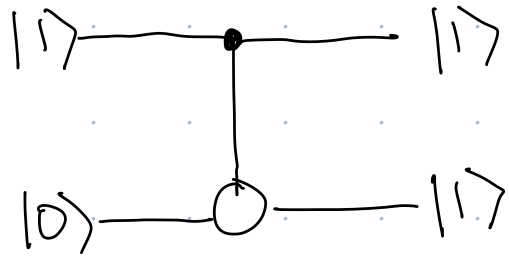
Quantum Gates (Circuit picture)

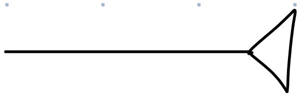
- time L → R
- wires are qubits
- gates + measurements are shapes



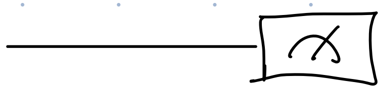
Initialize qubit in $|0\rangle$





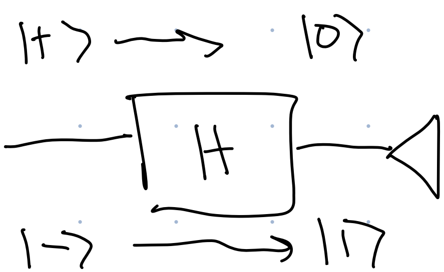


or



Measure in standard basis $M = \{|0\rangle, |1\rangle\}$

Measure in Hadamard



$\approx \quad M = \{|+\rangle, |-\rangle\}$

Need gate for f. What about:

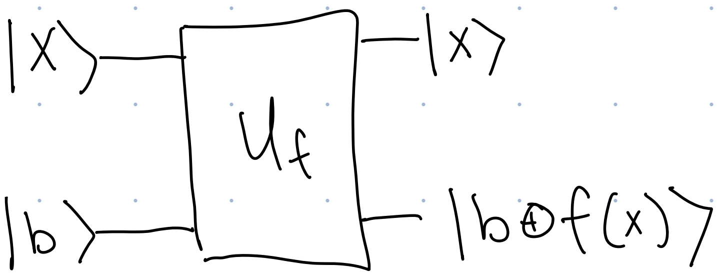
$|0\rangle \to |f(0)\rangle$ $\quad$ $|0\rangle \to |0\rangle$

$|1\rangle \to |f(1)\rangle$ $\quad$ $|1\rangle \to |0\rangle$

$f(0) = f(1) = 0$

Explain why this is not an allowed gate.
(State → State, Reversible)

Instead:



$|x\rangle|b\rangle \longrightarrow |x\rangle|b \oplus f(x)\rangle$

To interpret, $x, b \in \{0,1\}$
(standard basis states)

Interpretation:

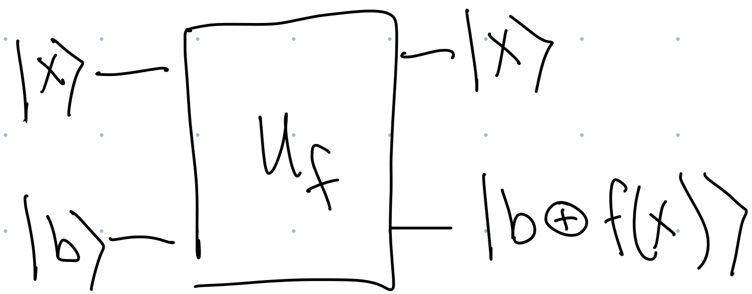
$|0\rangle|0\rangle \to |0\rangle|0 + f(0) \bmod 2\rangle$

$|0\rangle|1\rangle \to |0\rangle|1 + f(0) \bmod 2\rangle$

$|1\rangle|0\rangle \to |1\rangle|0 + f(1) \bmod 2\rangle$

$|1\rangle|1\rangle \to |1\rangle|1 + f(1) \bmod 2\rangle$

$U_f U_f = I$

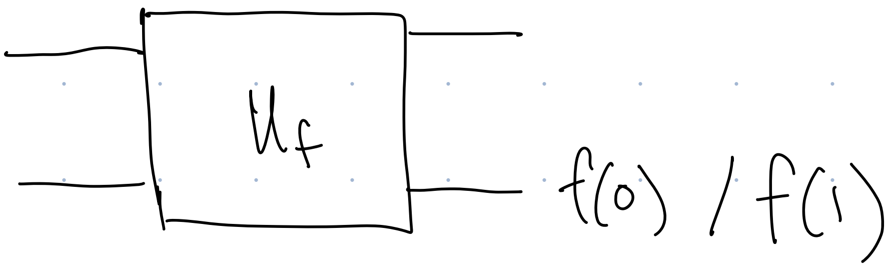


If $x=0$ get info about $f(0)$

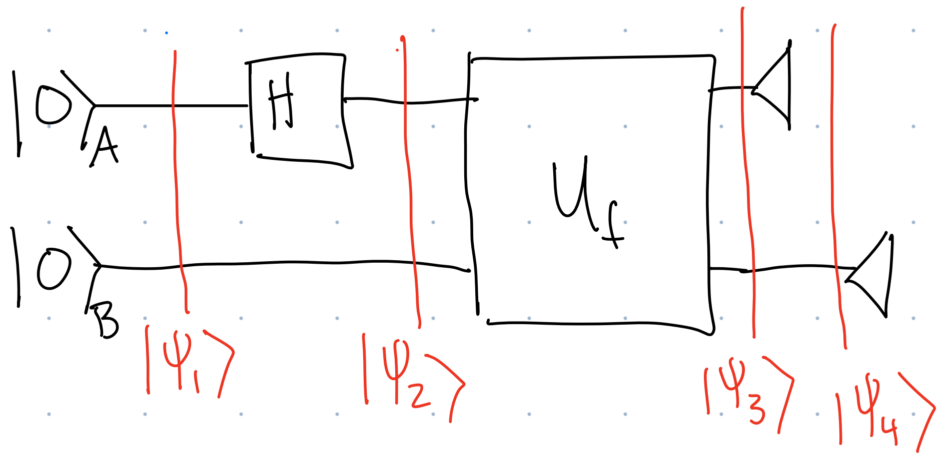
If $x=1$ get info about $f(1)$

Idea – Superposition!

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



Analyzing a Quantum Circuit



U_f:

$$\underbrace{|x\rangle|b\rangle \to |x\rangle|b \oplus f(x)\rangle}$$

ONLY WORK IF $|x\rangle, |b\rangle$ are standard basis states!!!

$|\psi_1\rangle = |0\rangle_A \otimes |0\rangle_B = |0\rangle_A|0\rangle_B$ (first qubit is top qubit)

$|\psi_2\rangle = H \otimes I\,|0\rangle|0\rangle = (H|0\rangle) \otimes I|0\rangle = |+\rangle|0\rangle$

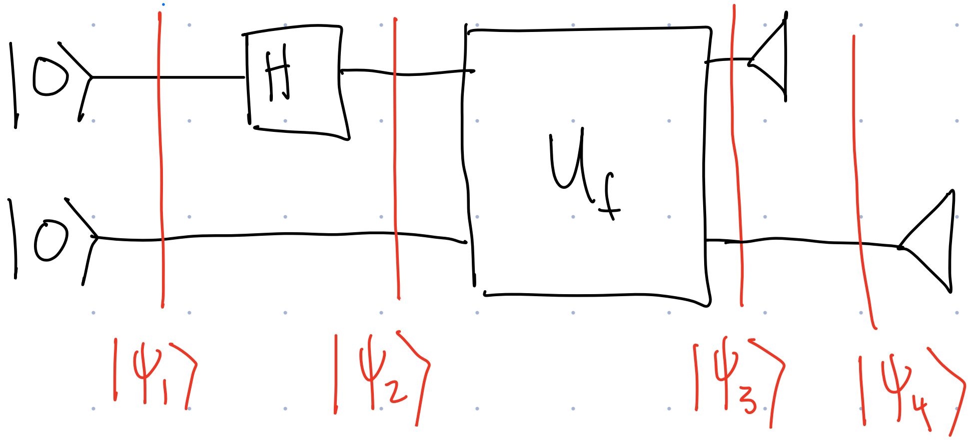
$|\psi_3\rangle = U_f|+\rangle|0\rangle$ ~~$= |+\rangle|0 \oplus f(+)\rangle$~~

$= U_f \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}} U_f|0\rangle|0\rangle + \frac{1}{\sqrt{2}} U_f|1\rangle|0\rangle$ $(x=1 \quad b=0)$

$(|1\rangle|0 \oplus f(1)\rangle)$

$= \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle$

$=$



$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle$$

Partial Measurement:

Outcome $|0\rangle$

- Prob $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- Collapse : $|0\rangle|f(0)\rangle \rightarrow |f(0)\rangle$

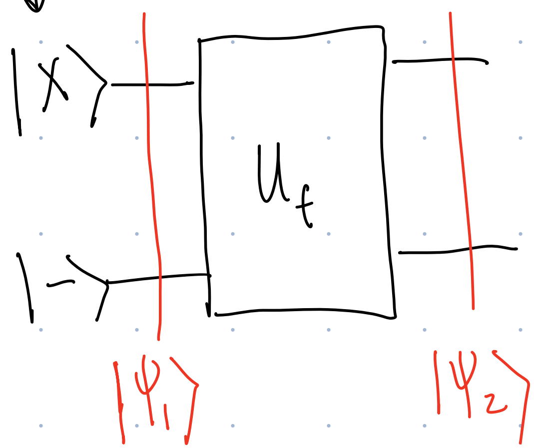
Outcome $|1\rangle$

- Prob $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$
- Collapse $|1\rangle|f(1)\rangle \rightarrow |f(1)\rangle$

Mini-Hype Lesson

Group Exercise:

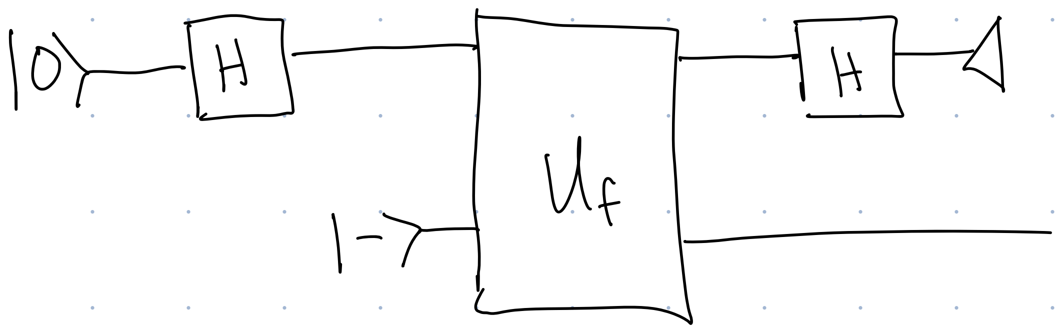
$x = 0$ or 1



$$|x\rangle|b\rangle \rightarrow |x\rangle|b \oplus f(x)\rangle$$

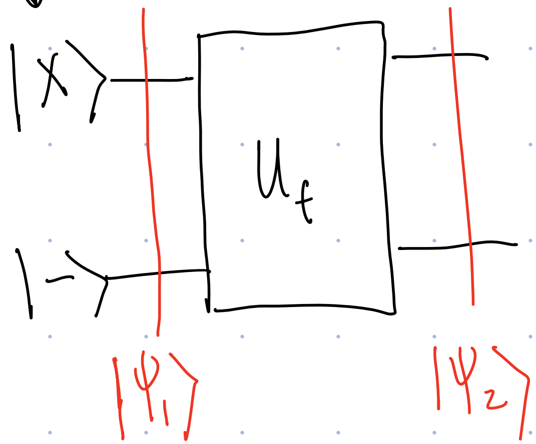
Show that $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|-\rangle$

Analyze outcome of this circuit:



Solution

$x = 0$ or 1



S.B.: $U_f|x\rangle|b\rangle = |x\rangle|b \oplus f(x)\rangle$

Show that $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|-\rangle$

$$|\psi_2\rangle = U_f|x\rangle|-\rangle$$
$$= U_f|x\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$= \frac{1}{\sqrt{2}}U_f|x\rangle|0\rangle - \frac{1}{\sqrt{2}}U_f|x\rangle|1\rangle$$
$$= \frac{1}{\sqrt{2}}|x\rangle|f(x)\rangle - \frac{1}{\sqrt{2}}|x\rangle|1 \oplus f(x)\rangle$$

$f(x) = 0$

$$\frac{1}{\sqrt{2}}|x\rangle|0\rangle - \frac{1}{\sqrt{2}}|x\rangle|1\rangle$$
$$= \frac{1}{\sqrt{2}}|x\rangle(|0\rangle - |1\rangle)$$
$$= |x\rangle|-\rangle$$

$f(x) = 1$

$$\frac{1}{\sqrt{2}}|x\rangle|1\rangle - \frac{1}{\sqrt{2}}|x\rangle|0\rangle$$
$$= \frac{1}{\sqrt{2}}|x\rangle(|1\rangle - |0\rangle)$$
$$= -|x\rangle|-\rangle$$

$$(-1)^{f(x)}|x\rangle|-\rangle$$

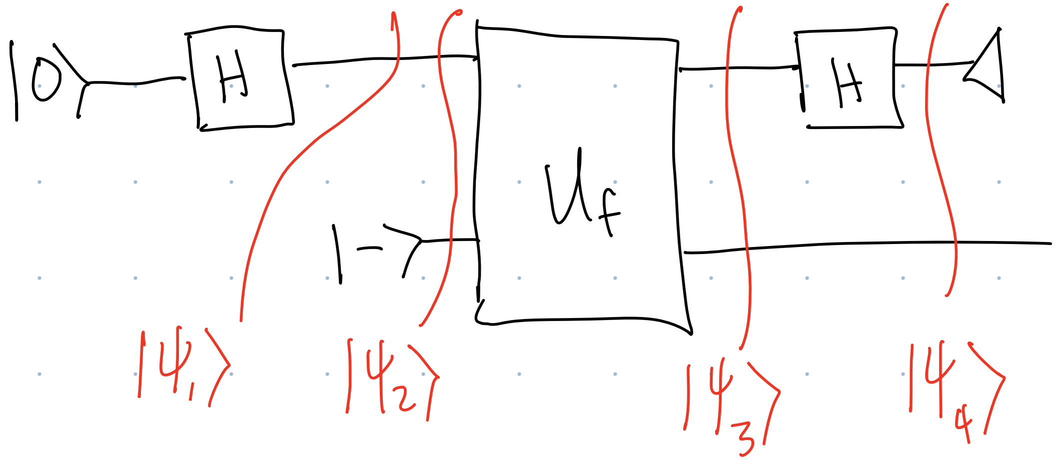
New rule for U_f:

$$\boxed{|x\rangle|-\rangle \to (-1)^{f(x)}|x\rangle|-\rangle}$$

standard basis state

Phase Kickback

U_f :



$|0\rangle \to i|0\rangle$

$|1\rangle \to i|1\rangle$

$$\boxed{|x\rangle|-\rangle \to (-1)^{f(x)}|x\rangle|-\rangle}$$

$|0\rangle|-\rangle + |1\rangle|-\rangle \to$

$(-1)^{f(0)}|0\rangle|-\rangle + (-1)^{f(1)}|1\rangle|-\rangle$

$|\psi_1\rangle = H|0\rangle = |+\rangle$

$|\psi_2\rangle = |+\rangle|-\rangle$

$|\psi_3\rangle = U_f|+\rangle|-\rangle$

$= U_f\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|-\rangle = \frac{1}{\sqrt{2}}U_f|0\rangle|-\rangle + \frac{1}{\sqrt{2}}U_f|1\rangle|-\rangle$

$= \frac{1}{\sqrt{2}}(-1)^{f(0)}|0\rangle|-\rangle + \frac{1}{\sqrt{2}}(-1)^{f(1)}|1\rangle|-\rangle$

$f(0)=f(1)$ flat

$= \frac{1}{\sqrt{2}}(-1)^{f(0)}\left(|0\rangle + |1\rangle\right)|-\rangle$

$= (-1)^{f(0)}|+\rangle|-\rangle$

$f(0)\neq f(1)$ sloped

$= \frac{1}{\sqrt{2}}(-1)^{f(0)}\left(|0\rangle + \boxed{(-1)^{f(0)}(-1)^{f(1)}}|1\rangle\right)|-\rangle$

$(-1)^{f(0)+f(1)}$

$= \frac{1}{\sqrt{2}}(-1)^{f(0)}\left(|0\rangle - |1\rangle\right)|-\rangle$

$$\Big\downarrow H \otimes I$$

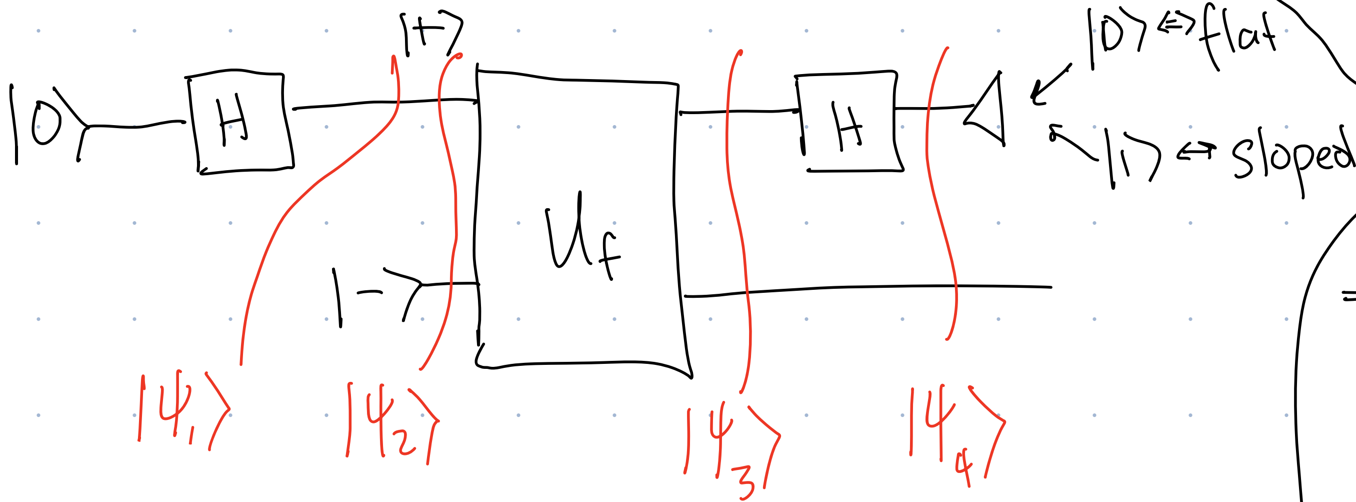
$$= (-1)^{f(0)} \big(H|+\rangle\big)|-\rangle$$

$$= (-1)^{f(0)} |0\rangle|-\rangle$$

$$= |0\rangle \left(\frac{(-1)^{f(0)}}{\sqrt{2}} |0\rangle - \frac{(-1)^{f(0)}}{\sqrt{2}} |1\rangle \right)$$

Prob of $|0\rangle$: $\left|\frac{(-1)^{f(0)}}{\sqrt{2}}\right|^2 + \left|\frac{(-1)^{f(0)}}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1$

Deutsch's Alg:



$$= (-1)^{f(0)} |-\rangle|-\rangle$$

$$\Big\downarrow H \otimes I$$

$$= (-1)^{f(0)} \big(H|-\rangle\big)|-\rangle$$

$$= (-1)^{f(0)} |1\rangle|-\rangle$$

Prob of $|1\rangle$: 1

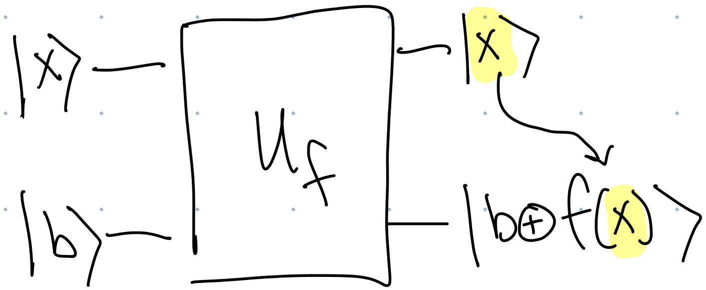
$$= \frac{1}{\sqrt{2}} (-1)^{f(0)} \left(|0\rangle + \boxed{(-1)^{f(0)} (-1)^{f(1)}} |1\rangle \right)|-\rangle$$

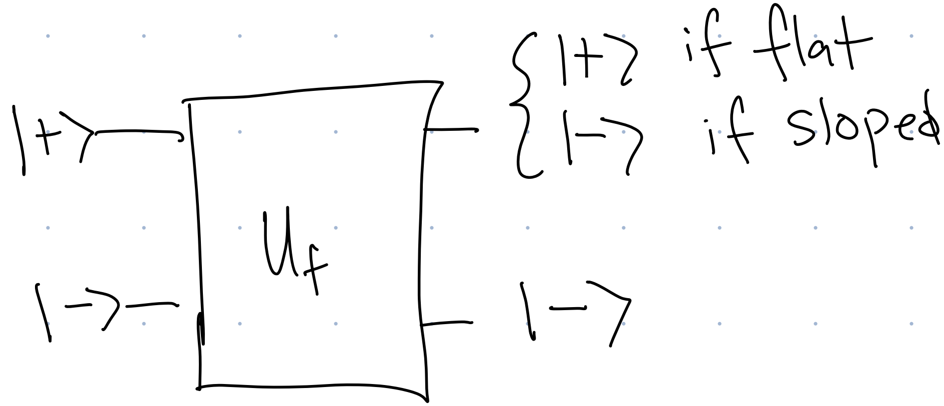
$$= \frac{1}{\sqrt{2}} \left((-1)^{f(0)} |0\rangle|-\rangle + \underbrace{\boxed{(-1)^{f(0)} (-1)^{f(0)}}}_{1} (-1)^{f(1)} |1\rangle|-\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left((-1)^{f(0)} |0\rangle|-\rangle + (-1)^{f(1)} |1\rangle|-\rangle \right)$$

Phase Kickback

Standard Basis





$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$+$ or $-$

$|+\rangle$ vs $|-\rangle$