Learning Goals

- Analyze 2-qubit systems
- Show a quantum advantage in game playing.

Announcements

- OH: Th 3:30-4:30 Fr: 11-12

Exit Tickets

Bra-ket = dot product + complex conjugation

Math rules

- $|\psi_1\rangle_A |\psi_2\rangle_B \xrightarrow{\text{Bra}} \langle\psi_1|_A \langle\psi_2|_B$
- $\langle\psi|_B |\phi\rangle_A \rightarrow |\phi\rangle_A \langle\psi|_B$

More qubits = More better?

How much work to show

CHSH Game (Clauser, Horne, Shimony, Holt) (Bell Inequality)

Time

$$\begin{tikzcd}
 & \text{Referee} \arrow[dl, "x"'] \arrow[dr, "y"] & \\
\text{Amir} \arrow[r] \arrow[dr, "a"'] & \text{No communication} & \text{Bei} \arrow[l] \arrow[dl, "b"] \\
 & \text{Referee} &
\end{tikzcd}$$

$x, y, a, b \in \underline{\underline{\{0,1\}}}$

Amir + Bei win if $a \oplus b = x \wedge y$

+ mod 2 ($\oplus$)

logical AND ($\wedge$)

Best Strategy

$a = b = 0$

$a \oplus b = 0$

$x \wedge y = 0$ 75% of time

Q. What is Amir + Bei's best strategy, if referee chooses x, y randomly? What is their probability of winning? 75%

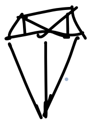
Q: Hype podcast → still excited about q?

What if we give them each a qubit?

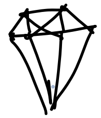
Diamond nitrogen vacancy qubit

~ 1 sec coherence

Amir

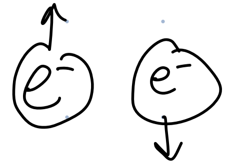


Bei



$|0\rangle$

$|1\rangle$



$|0\rangle$ 

$|1\rangle$ 

How do we represent the combined state of both qubits?

Tensor Product

A: $|0\rangle \otimes |1\rangle$ ✓

B: $|0\rangle_A |1\rangle_B$ ✓

C: $|01\rangle_{AB}$ ✓

D: $|01\rangle$ ✓

☆ Bra + ket

$\langle 0| \rightarrow \leftarrow |1\rangle = \langle 0|1\rangle \rightarrow$ number

$\langle 0| |0\rangle \rightarrow 1$

Ket + ket

$|0\rangle \rightarrow |1\rangle \rightarrow |01\rangle = |0\rangle|1\rangle$

Qubit A

$|\psi_1\rangle_A = a_0|0\rangle + a_1|1\rangle$

Qubit B

$|\psi_2\rangle_B = b_0|0\rangle + b_1|1\rangle$

Combined: $|\psi\rangle_{AB} = |\psi_1\rangle_A \otimes |\psi_2\rangle_B = |\psi_1\rangle_A |\psi_1\rangle_B$

$= (a_0|0\rangle + a_1|1\rangle)_A \otimes (b_0|0\rangle + b_1|1\rangle)_B$

Distribute $\otimes$ multiplication

$= a_0 b_0 |0\rangle_A |0\rangle_B + a_0 b_1 |0\rangle|1\rangle$

$+ a_1 b_0 |1\rangle_A |0\rangle_B + a_1 b_1 |1\rangle|1\rangle$

Scalars commutes

Keep ket order

$= a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle$

Generic 2 Qubit State:

$|\psi\rangle_{AB} = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$

"amplitudes"

$a_{00}, a_{01}, a_{10}, a_{11} \in \mathbb{C}$

complex numbers

$$\sum_{i \in \{0,1\}^2} |a_i|^2 = 1$$

Vectors

$$\begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix}$$

Amir + Bei's Strategy

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Amir Receives	Amir does measurement
$x=0$	$M(0)$
$x=1$	$M\left(\frac{\pi}{4}\right)$

$\rightarrow R \rightarrow a=0$

$\rightarrow Q \rightarrow a=1$

Bei Receives	Bei does measurement
$y=0$	$M\left(\frac{\pi}{8}\right)$
$y=1$	$M\left(\frac{-\pi}{8}\right)$

$\rightarrow R \rightarrow b=0$

$\rightarrow Q \rightarrow b=1$

$$M(w) = \{ \overset{R}{\cos(w)|0\rangle + \sin(w)|1\rangle}, \ \overset{Q}{-\sin(w)|0\rangle + \cos(w)|1\rangle} \}$$

$$M(0) = \{ \cos(0)|0\rangle + \sin(0)|1\rangle, \ -\sin(0)|0\rangle + \cos(0)|1\rangle$$

$$= \{ |0\rangle, |1\rangle \}$$

If Amir measures $M_A = \{ |\phi_{1a}\rangle, |\phi_{2a}\rangle \}$ and
Bei measures $M_B = \{ |\phi_{1b}\rangle, |\phi_{2b}\rangle \}$ on their combined

state $|\psi\rangle_{AB}$, probability of outcome $\underline{|\phi_{ia}\rangle_A \, |\phi_{jb}\rangle_B}$ is
(according to the laws of physics)

$|\underline{\langle\phi_{ia}|_A \langle\phi_{jb}|_B} \, \underline{|\psi\rangle}_{AB}|^2$, and $|\psi\rangle_{AB}$ collapses to $\underline{|\phi_{ia}\rangle_A |\phi_{jb}\rangle_B}$

DO NOT

SWITCH ORDER!

$\cancel{\langle\phi_{jb}|_B \langle\phi_{ia}|_A}$

Example

What is the probability of outcome $|R(\frac{\pi}{4})\rangle_A , |Q(\frac{-\pi}{8})\rangle_B$?

$$|R(\tfrac{\pi}{4})\rangle \otimes |Q(\tfrac{-\pi}{8})\rangle$$

$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$

$$\left| \langle R(\tfrac{\pi}{4})|_A \otimes \langle Q(\tfrac{-\pi}{8})|_B \, |\psi\rangle_{AB} \right|^2$$

$$= \left| \left(\cos\tfrac{\pi}{4}\langle 0|_A + \sin\tfrac{\pi}{4}\langle 1|_A\right) \otimes \left(\overset{+}{-}\sin\left(\tfrac{\overset{+}{-}\pi}{8}\right)\langle 0|_B + \cos\left(\tfrac{\overset{+}{-}\pi}{8}\right)\langle 1|\right)\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\right)\right|^2$$

$$= \left|\left(\cos\tfrac{\pi}{4}\sin\left(\tfrac{\pi}{8}\right)\langle 00|_{AB} + \cos\tfrac{\pi}{4}\cos\left(\tfrac{\pi}{8}\right)\langle 01|_{AB} + \sin\tfrac{\pi}{4}\sin\tfrac{\pi}{8}\langle 10|_{AB} + \sin\tfrac{\pi}{4}\cos\tfrac{\pi}{8}\langle 11|_{AB}\right)\left(\tfrac{1}{\sqrt{2}}|00\rangle_{AB} + \tfrac{1}{\sqrt{2}}|11\rangle_{AB}\right)\right|^2$$

$$= \left| \boxed{\tfrac{1}{\sqrt{2}}\cos\tfrac{\pi}{4}\sin\tfrac{\pi}{8}} \overset{1}{\langle 00|00\rangle_{AB}} + \cancel{\tfrac{1}{\sqrt{2}}\cos\tfrac{\pi}{4}\sin\tfrac{\pi}{8}\overset{0}{\langle 00|11\rangle}} + \tfrac{1}{\sqrt{2}}\cos\tfrac{\pi}{4}\cos\tfrac{\pi}{8}\overset{0}{\langle 01|00\rangle} \right.$$

$$\langle 01|_{AB}|00\rangle_{AB} = \left(\langle 0|_A\langle 1|_B\right)\left(|0\rangle_A|0\rangle_B\right) = \overset{1}{\boxed{\langle 0|_A|0\rangle_A}}\,\overset{0}{\boxed{\langle 1|_B|0\rangle_B}} = 0$$

$$= \left| \frac{1}{\sqrt{2}} \cos\frac{\pi}{4} \sin\frac{\pi}{8} + \frac{1}{\sqrt{2}} \sin\frac{\pi}{4} \cos\frac{\pi}{8} \right|^2 \approx .426$$

S

Question	Winning Outcomes	Probability
① $X=0, y=0$ $X \wedge y = 0$	$\|R(0)\rangle \|R(\frac{\pi}{8})\rangle$ or $\|Q(0)\rangle \|Q(\frac{\pi}{8})\rangle$	$.426$ $.426$ $\}\approx .85$
② $X=0, y=1$ $X \wedge y = 0$	$\|R(0)\rangle \|R(\frac{-\pi}{8})\rangle$ $\|Q(0)\rangle \|Q(\frac{-\pi}{8})\rangle$	$.426$ $.426$ $\}\approx .85$
$X=1, y=1$ $X \wedge y = 1$	$\|R(\frac{\pi}{4})\rangle, \|Q(\frac{-\pi}{8})\rangle$ $a=0$ $b=1$	$\rightarrow .426$ $+$ $\}\approx .85$

$X \wedge y = a \oplus b = 1$ $\quad |Q(\frac{\pi}{4})\rangle, |R(\frac{-\pi}{8})\rangle \rightarrow .426$

$a=1$ $\quad b=0$

$a=0, b=1$

$a=1, b=0$ $\quad a \oplus b = 1$

③ $X=1$ $y=0$ $\quad |R(\frac{\pi}{4})\rangle |R(\frac{\pi}{8})\rangle$

$|Q(\frac{\pi}{4})\rangle |Q(\frac{\pi}{8})\rangle$

$.426$

$.426$ $\}\approx .85$

$.85 > .75$

- Determine winning outcomes
- Divide 2 winning outcomes + calculate prob of that outcome.
- Check calc with group
- Add to get total prob of success

For all of referees questions, success prob is

Why play CHSH game?

- Verify quantumness
- Generate genuine randomness