Learning Goals

- Define NP-complete and NP-Hard Problems and describe their importance
- Prove a problem is NP-Hard [NP2]

Types of Problems

Easy
(Polynomial time)

- Search
- Sort
- Multiplication
- Closest Points
- Greedy Scheduling
- MWIS on a line
- Matrix Mult.

Puzzles / NP

Crossword

Sudoku

Delivery rt ≤ 100 miles

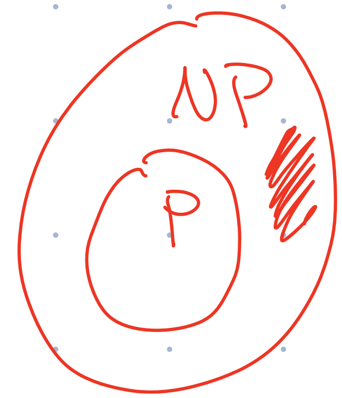
Protein Folding

Factor larger numbers

Primality Testing

Quantum

Reg.



Question: How do we identify the hardest problems in NP?

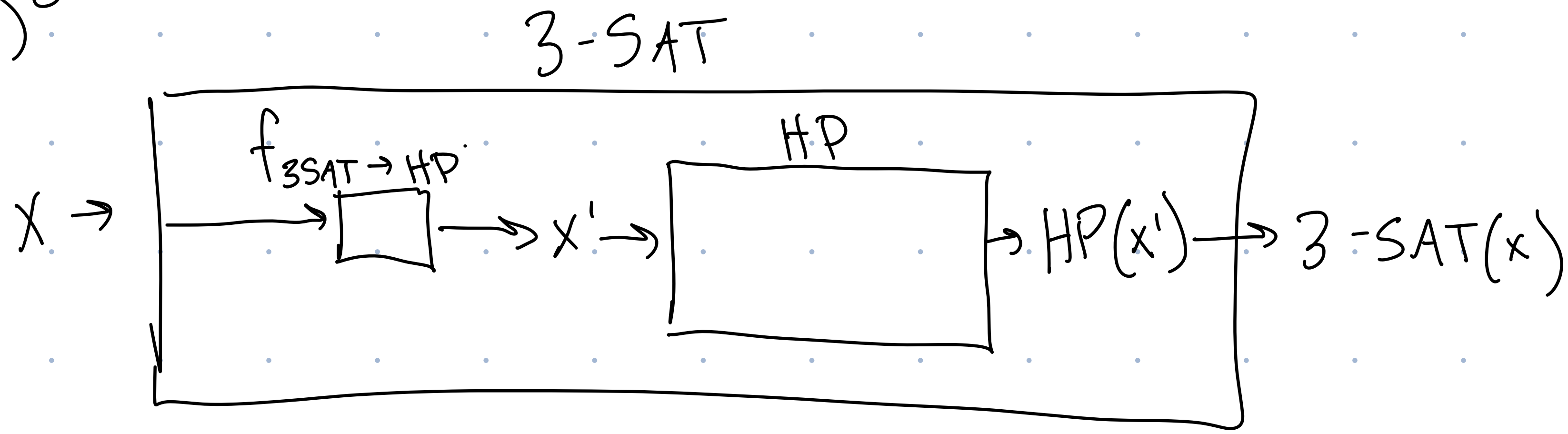
→ Empirical: If keep trying to find an alg, but can't...

→ Analytical: Can we <u>prove</u> a problem is hard? ☆

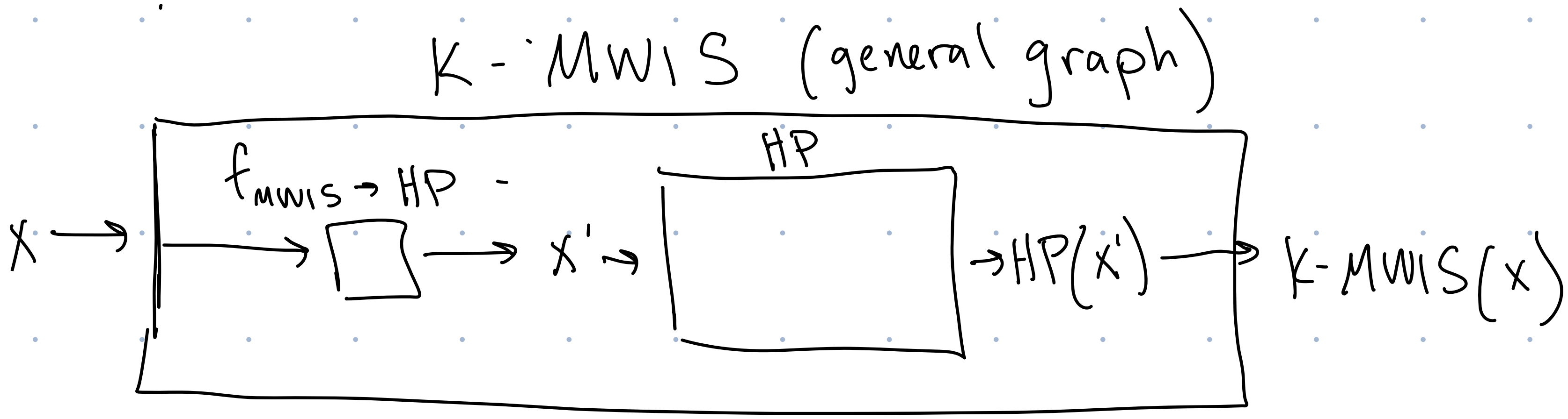
NP-Hard

A problem $Q \in$ NP-Hard if for every problem $R \in NP$, $R \leq_p Q$

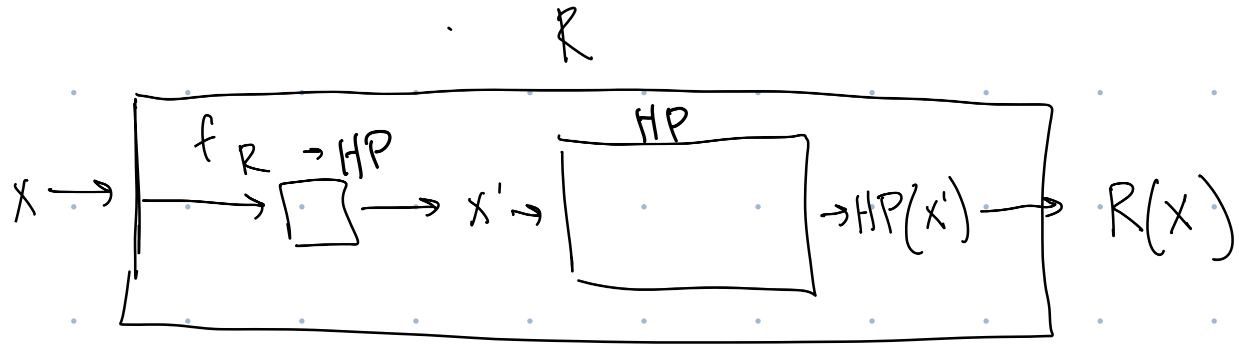
Ex: Halting Problem (HP) $\in$ NP-Hard so



Also

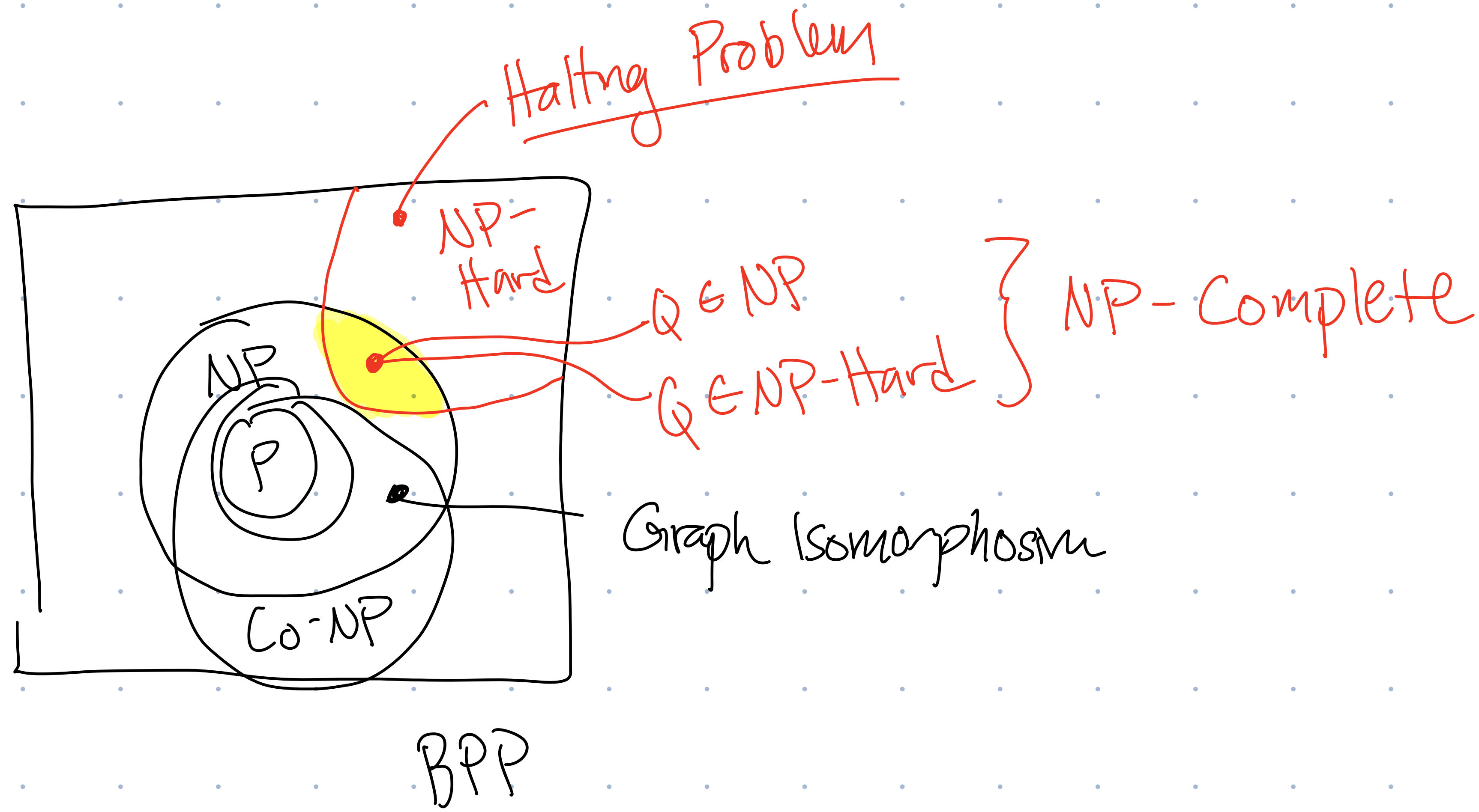


Also For any $R \in NP$, $\exists$ $f_{R \to HamPath}$



→ NP-Hard problems are harder / require more resources than NP problems, b/c give power to solve all in NP

BUT



def: Q ∈ NP-Complete if Q ∈ NP and Q ∈ NP-Hard

$\boxed{\text{Fact 1: } 3SAT \in NP\text{-Hard}}$ (see 301)

Lemma 1: $\overset{3\text{-SAT}}{R} \in NP\text{-Hard}$ and $\overset{3\text{-SAT} \leq_p \text{HamPath}}{R \leq_p Q}$ then $\overset{\text{HamPath}}{Q} \in NP\text{-Hard}$.

NP-Hard Bootstrapping Lemma (See pset 9)

Theorem: Ham-Path is NP-Complete

Pf
- Ham-Path $\in$ NP [insert proof here... see NP class]
- Ham-Path $\in$ NP-Hard

Will Prove: $3SAT \leq_p$ HamPath

\+

Fact 1

\+

NP-Hard Bootstrapping Lemma

Formal Definition of Polytime Reduction

$\{0,1\}^* =$ set of all bitstrings

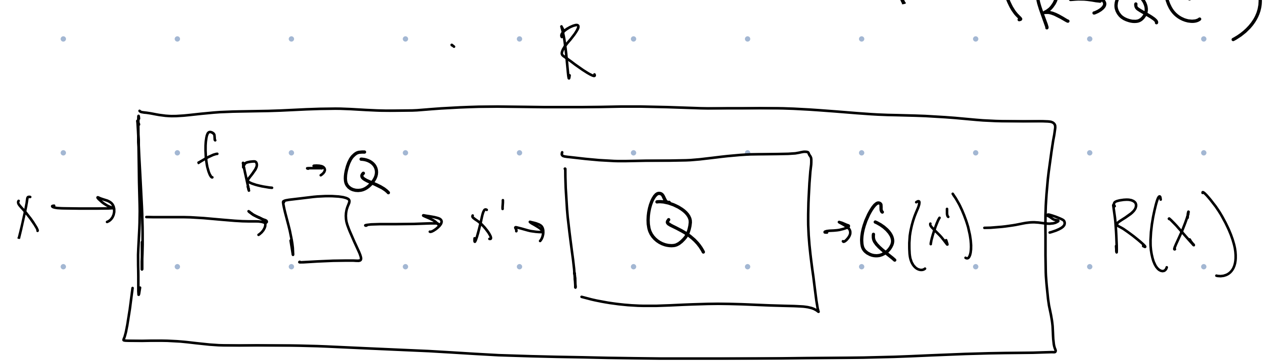
<u>def</u>: $R \leq_p Q$ (R is polytime reducible to Q) if

$\exists\, f_{R\to Q} : \{0,1\}^* \to \{0,1\}^*$, s.t. 1

- $\forall x \in \{0,1\}^*$, $R(x) = \text{Yes}$ iff $Q(f_{R\to Q}(x)) = \text{Yes}$ 2
- $\exists$ constant $c_{R\to Q}$ s.t. runtime of $f_{R\to Q}$ on input x is $O(|x|^{c_{R\to Q}})$ 3

$f_{R\to Q}$ runs in polynomial time

$x' = f_{R\to Q}(x)$



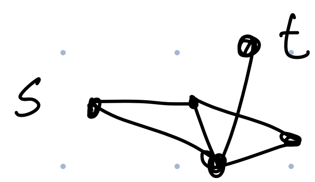
Lemma: 3SAT $\le_p$ Ham-Path

Strategy: 1. Describe $f_{3SAT \to Ham\text{-}Path}$ (turn 3SAT $\to$ Ham-Path)

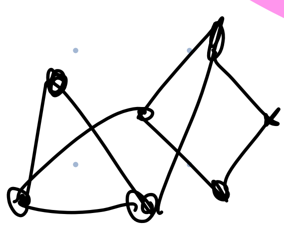
2. Show X is 3SAT-Yes iff $f_{3SAT \to HP}(X)$ is Ham Path-Yes

3. Show $f_{3SAT\text{-}Ham\text{-}Path}$ polytime

1. $(z_1 \vee \neg z_2 \vee z_3) \wedge \cdots \Longrightarrow$

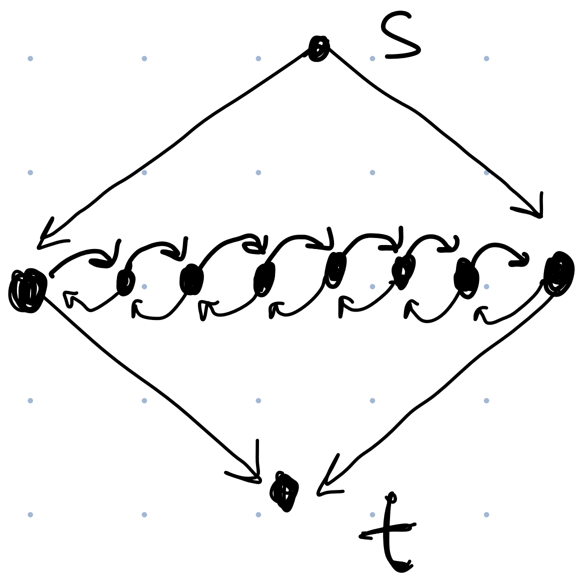


Common mistake!



$\Longrightarrow (z_2 \vee z_6 \vee \neg z_8) \wedge \cdots$

How many Hamiltonian Paths are in this graph?



A. 2

B. 3

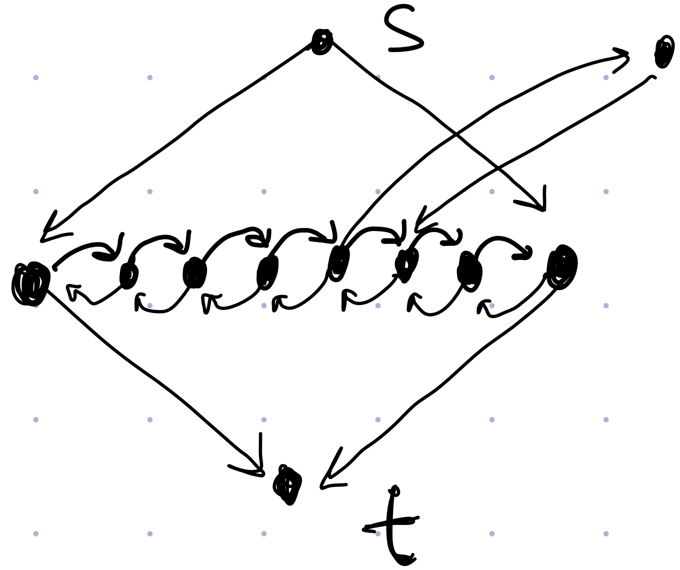
C. 49

D. $\binom{7}{2}$

How many Hamiltonian Paths are in this graph?

LRL

True

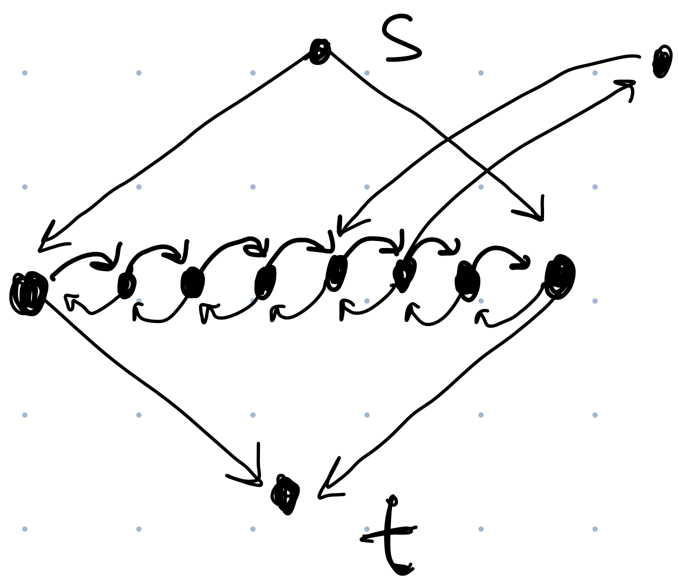


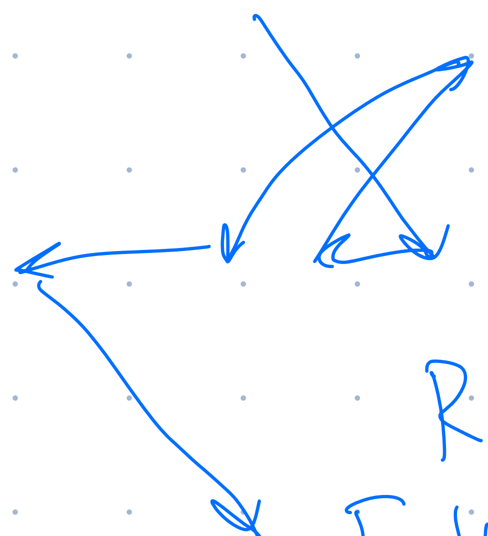
A. 0

B. 1

C. 2

D. 3

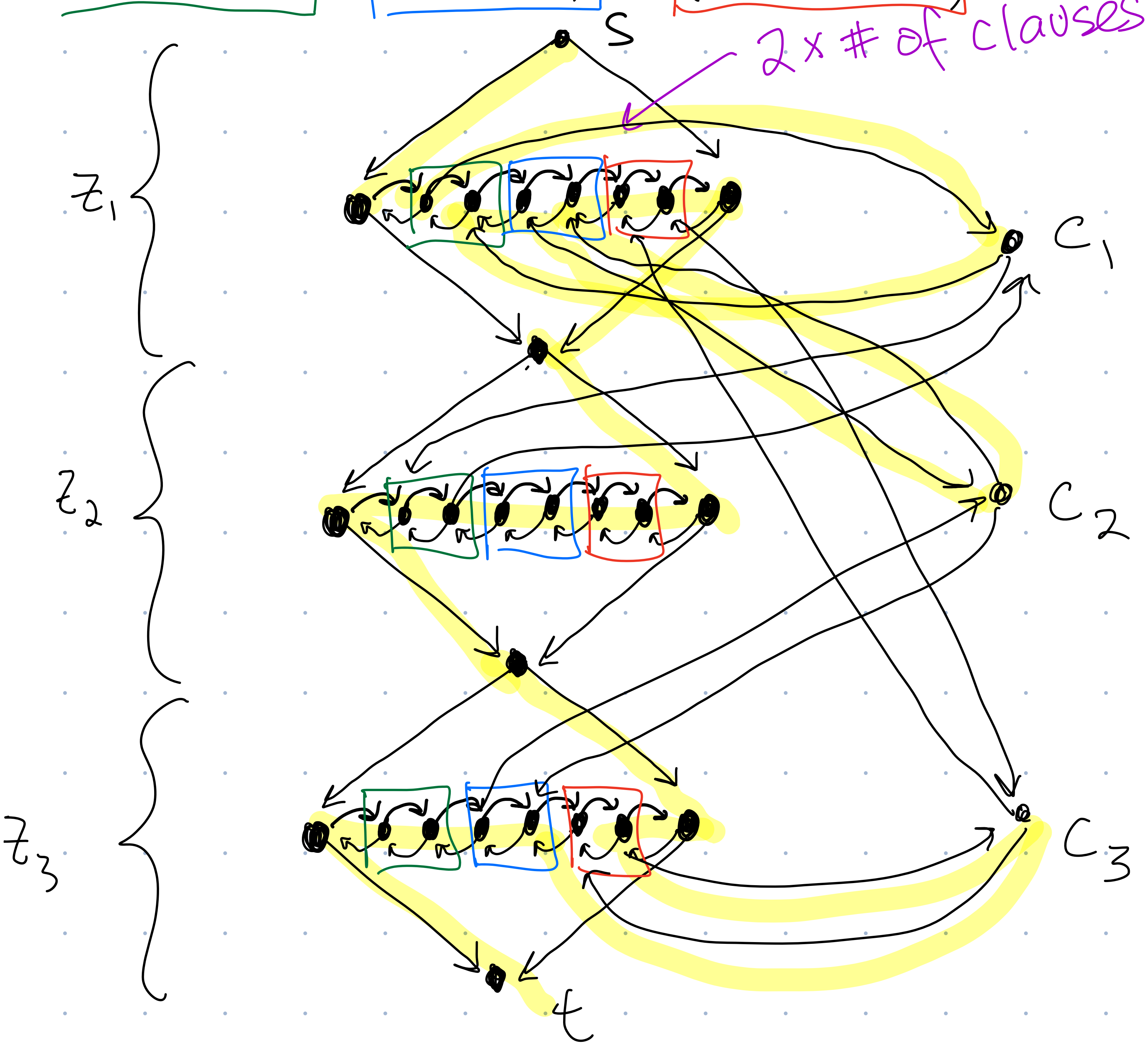




RLR

False

$X = (z_1 \vee \neg z_2) \wedge (z_1 \vee z_3) \wedge (\neg z_1 \vee \neg z_3)$



TRUE

False

$z_1 = T$

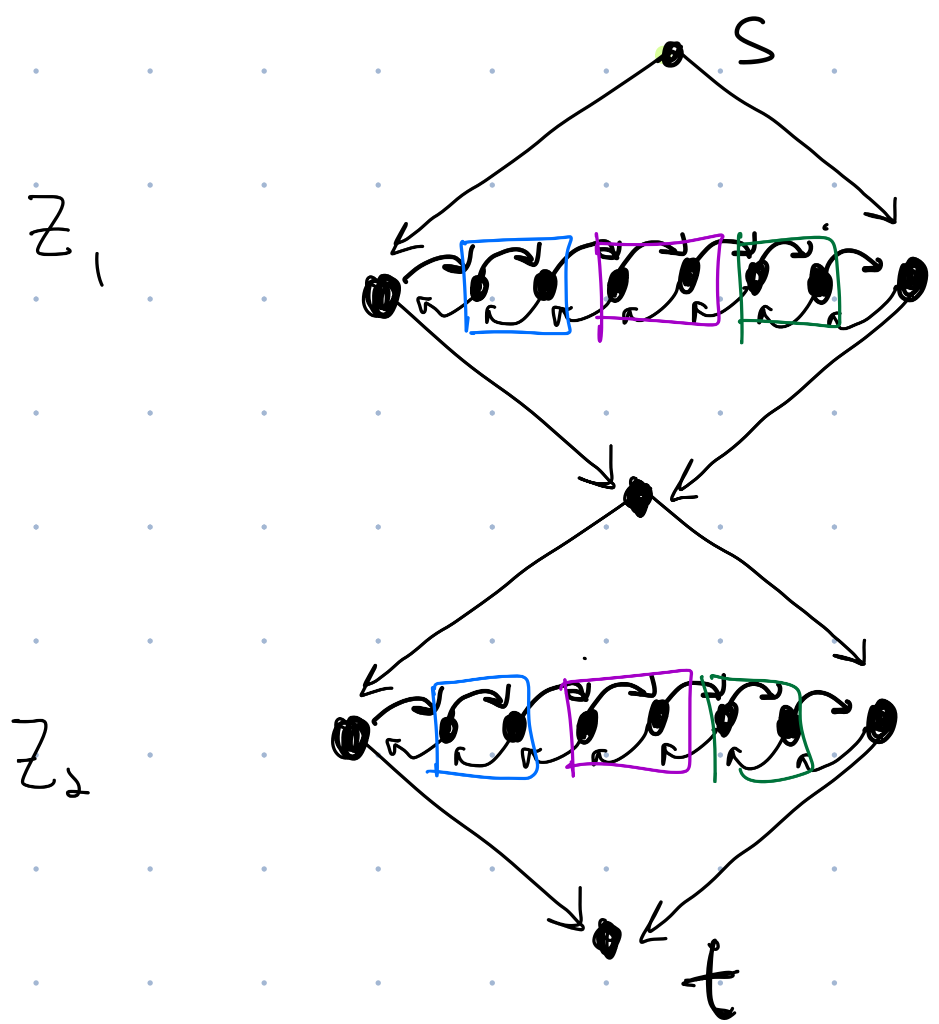
$z_2 = F$

$z_3 = F$

Group Work

1. Encode $(z_1) \wedge (\neg z_1 \vee z_2) \wedge (\neg z_1 \vee \neg z_2)$ into Ham-Path instance. Show get a No Instance.
2. Runtime of $f_{3SAT \to HAM\text{-}PATH}$? (Create adj matrix for graph)
3. $3SAT(x) = Yes$ iff $HAMPATH(f_{3SAT-HAMPATH}(x)) = Yes$

1. $(z_1) \wedge (\neg z_1 \vee z_2) \wedge (\neg z_1 \vee \neg z_2)$



2.

$$3SAT(x) = Yes \quad \text{iff} \quad HAMPATH(f_{3SAT-HAMPATH}(x)) = Yes$$

$\Rightarrow$

$\Leftarrow$

3. Let $m = \#$ clauses
$n = \#$ variables

Each gadget:

Total gadgets:

Clause vertices:

Adjacency matrix

Total edges: Gadget:

Clauses:

Note: Once we prove Ham-Path $\in$ NP-Hard, we can combine with Lemma 1 to prove new problems are NP-Hard

Lemma 1: If $Q \in$ NP-Hard and $Q \le_p R$ then $R \in$ NP-Hard.

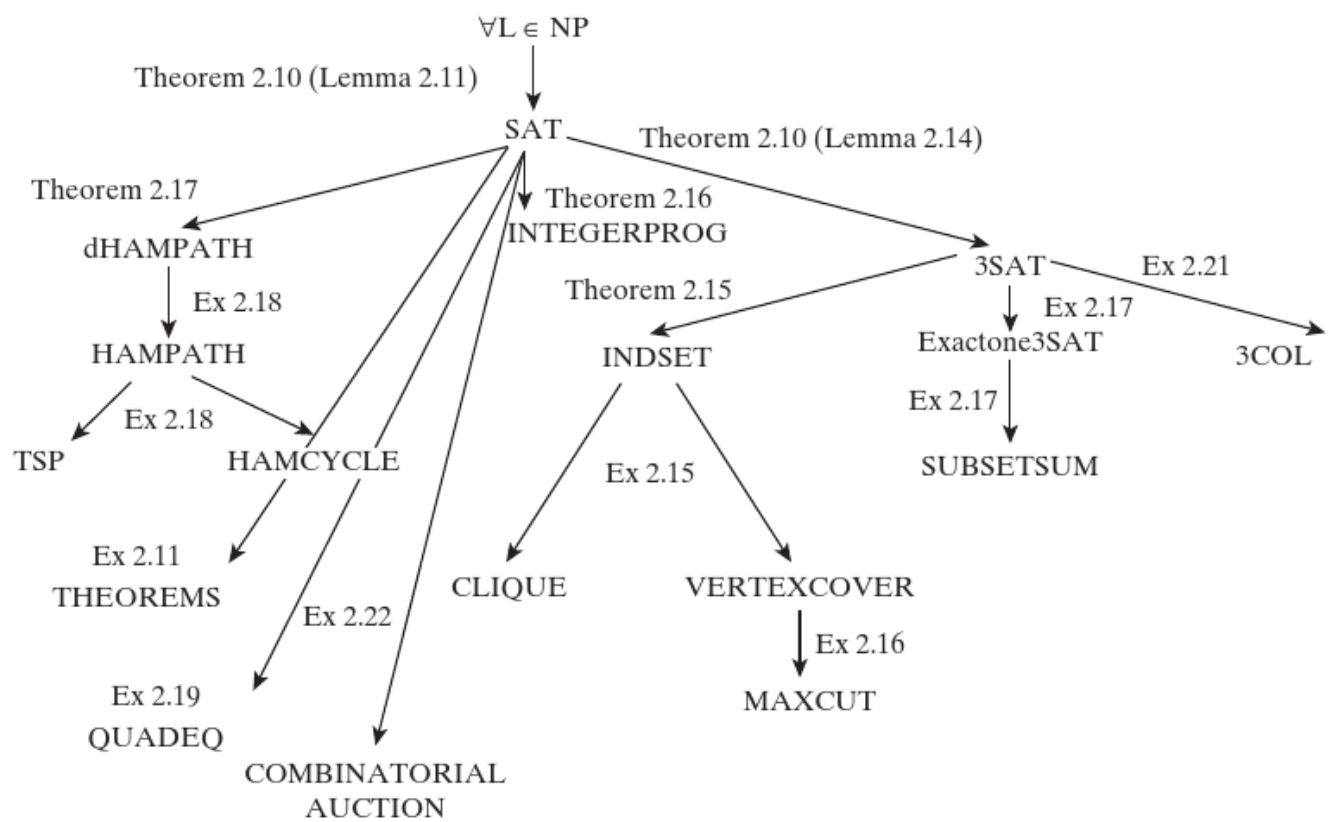


Figure 2.4. Web of reductions between the **NP**-completeness problems described in this chapter and the exercises. Thousands more are known.

(Arora + Boaz, Computational Complexity)