Bellman-Ford Algorithm

Learning Goals

- Design pseudocode for a dynamic programming alg. [DP2]
- PA2

Announcements

Thursday: Philosophy of AI 4:30, Twilight 201

Friday 4/10 CS Seminar on AI Audits

- 12:20-1:20 in 75 Shannon 102
- Indian Food

Exam - next week Thursday 4/16 7:30-9 pm
- 1 notecard (both sides)
- DP2, G2 (new)

Exit Tickets

NP, input size etc → OH

Shortest Path Problem

Input: $G = (V, E)$, $w : E \to \mathbb{R}$, $s, t \in V$, $|V| = n$, $|E| = m$, directed, no negative cycles

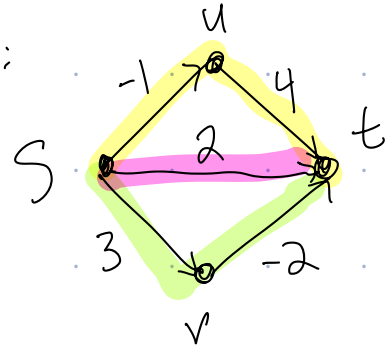
Output: Path P from s to t in G

e.g. $P = ((s,u), (u,v), \cdots (r,t))$ ← sequence of connected edges

such that $L(P) = \sum_{e \in P} w(e)$ is minimized

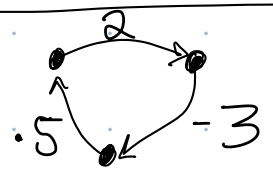
Example:

G:



Shortest Path: $P = ((s,v), (v,t))$

Negative Cycle:



Various Approaches

- Brute Force → Breadth-First Search
- Greedy → Dijkstra's
- Dynamic → Bellman-Ford

Limit

All edges wt 1

All wt are positive

- No neg. cycles
- Distributed graph

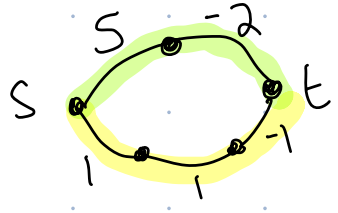
Applications:

- Data Routing in Networks
- Bartering
- Arbitrage

Defining Subproblems

Clever idea: $P_{u,i}$ = shortest path from s to u that uses at most i edges

What is $L(P_{t,1})$, $L(P_{t,2})$, $L(P_{t,3})$ for the following graph



A) ∞, 3, 1 B) ∞, 2, 3 C) 0, 2, 3 D) 0, 3, 1

Last choice a strategy makes? Last edge before t ↕ vertex

Designing a Dynamic Prog. Alg.

- Recurrence relation for $P_{u,i}$ = shortest path from s to u that uses at most i edges

$$P_{u,i} = \begin{cases} P_{v,i-1} + (v,u) & \text{if shortest path with } i \text{ edges goes through } v \text{ prior to } u \\ P_{w,i-1} + (w,u) & \text{if shortest path with } i \text{ edges goes through } w \text{ prior to } u \\ \vdots & \\ P_{u,i-1} & \text{if shortest path to } u \text{ uses less than } i \text{ edges} \end{cases}$$

- Recurrence relation for objective function $\begin{pmatrix} w(u,u) = 0 \\ w(u,v) = \infty \text{ if no edge} \end{pmatrix}$

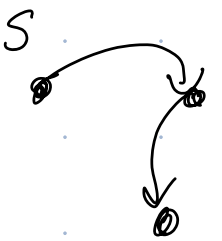
$$A(u,i) = \begin{cases} \min_{v \in V} \{ A(v, i-1) + w(v,u) \} \\ \text{Base case: } \begin{cases} A(s,i) = 0 \\ A(u,i) = \infty \quad \text{if } u \neq s \end{cases} \text{ if } i = 0 \end{cases}$$

Helpful notation $\rightarrow \min_{u \in S} \{ f(u) \}$

- Write Pseudocode to fill in A

Input: $n \times n$ array w where $w[u,v]$ = weight of edge (u,v), $s, t \in [n]$.
$w[u,v] = \infty$ if no edge. $w[u,u] = 0 \ \forall u \in [n]$

Output: ?x? array A s.t. $A[u,i] = L(P_{u,i})$



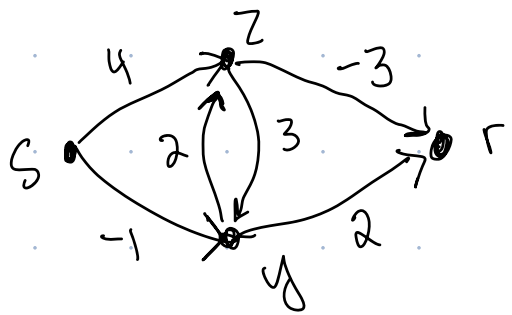
$n = 3$

Initialize $A \leftarrow n \times n$ fill in with ∞'s.

$A[s,0] \leftarrow 0$

```
for i ← 1 to n-1:
    for u ∈ V:
        for v ∈ V:
            if A[u,i] ≥ A[v,i-1] + w(v,u):
                A[u,i] ← A[v,i-1] + w(v,u)
```

Use your code to create A for the graph

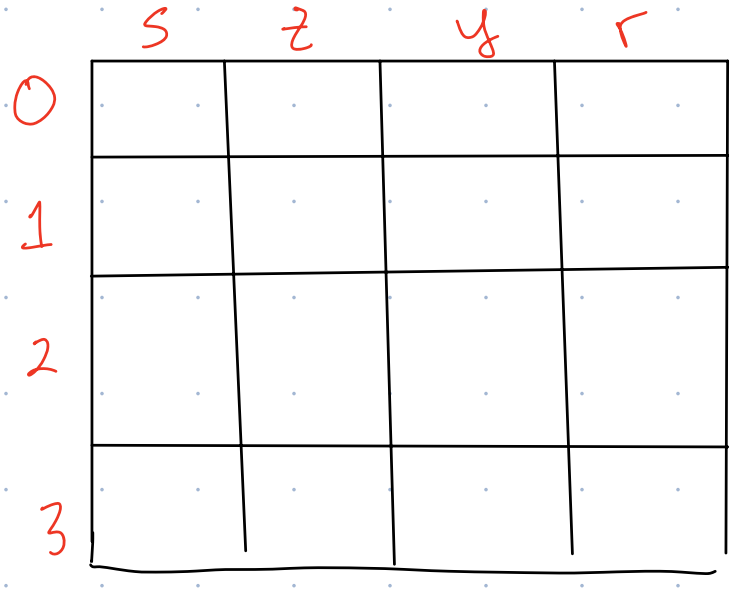


What is the Runtime of Bellman-Ford?

A) $O(n)$ B) $O(n^2)$ C) $O(n^3)$ D) $O(nm)$

n = # vertices, m = # edges

A



What is the Runtime of Bellman-Ford?

A) $O(n)$ B) $O(n^2)$ C) $O(n^3)$ D) $O(nm)$ (n = # vertices, m = # edges)

Recall

Recurrence relation for objective function ($w(u,u) = 0$, $w(u,v) = \infty$ if no edge)

$$L(P_{u,i}) = \begin{cases} \min\left\{ \min_{\substack{v \in V \\ v \neq u}} \left\{ L(P_{v,i-1}) + w(v,u) \right\}, \; L(P_{u,i-1}) \right\} \\ \text{Base case: } \begin{array}{l} L(P_{u,0}) = \infty \quad \text{if } u \neq s \\ L(P_{s,0}) = 0 \end{array} \end{cases}$$

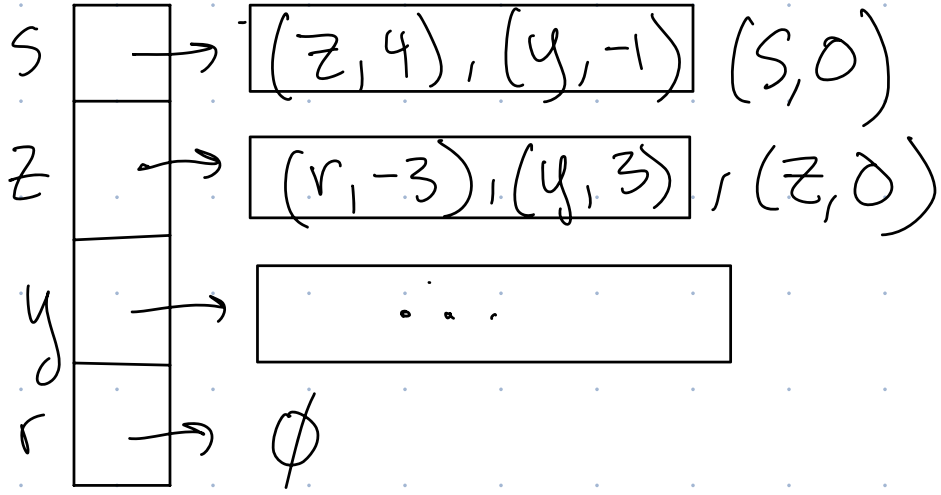
Don't want to check every $v \in V$. Only want to check $v : (v,u) \in E$.

Adjacency Matrix

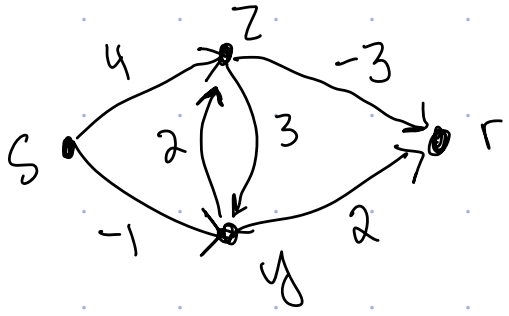
	s	z	y	r
s	0	4	-1	∞
z	∞	0	3	-3
y	∞	2	0	2
r	∞	∞	∞	0

s→r weight

Adjacency List



list of vertices with edges leaving a vertex



Reverse Adjacency List

list of vertices with edges entering a vertex

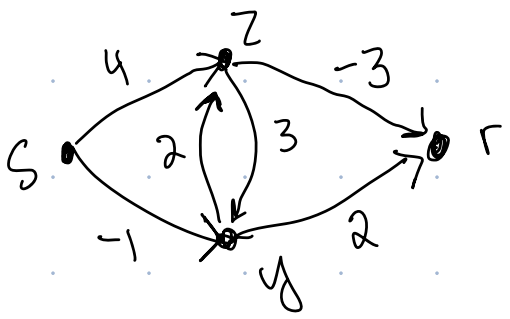
$w[s] = \emptyset$

$w[z] = [(s,4), (y,2)]$

$w[r] = [(z,-3), (y,2)]$

$w[y] = [(s,-1), (z,3)]$

e.g.



Initialize $A \leftarrow n \times n$ array filled with ∞'s.

$A[s,0] \leftarrow 0$

For $i \leftarrow 1$ to $n-1$:

  For $u \leftarrow 1$ to n:

    $A[u,i] = A[u,i-1]$

    For ~~$v \leftarrow 1$ to n~~: $v \in w[u]$ ← reverse adjacency list for u.

      If $A[u,i] > A[v,i-1] + w[v,u]$

        $A[u,i] \leftarrow A[v,i-1] + w[v,u]$

What is the Runtime of Bellman-Ford with Reverse Adj Matrix?

A) $O(n)$ B) $O(n^2)$ C) $O(n^3)$ D) $O(nm + n^2)$

(n = # vertices, m = # edges)

Initialize $A \leftarrow n \times n$ array filled with ∞'s.

$A[s,0] \leftarrow 0$

For $i \leftarrow 1$ to $n-1$:

  For $u \leftarrow 1$ to n:

    $A[u,i] = A[u,i-1]$

    ~~For $v \leftarrow 1$ to n~~ For $v : (v,u) \in E$:

      If $A[u,i] > A[v,i-1] + w[v,u]$

        $A[u,i] \leftarrow A[v,i-1] + w[v,u]$