Learning Goals

- Describe Knapsack problem ✓
- Develop recurrence for Knapsack ✓
- Create dynamic programming alg
- Analyze runtime

Announcements

Knapsack Problem

Input: n items
- $v_i \in \mathbb{N}$ is value of i^{th} item
- $w_i \in \mathbb{N}$ is weight of i^{th} item

Capacity: W (max weight) $\in \mathbb{N}$

Applications
- Shipping
- Exam

Output: $S \subseteq [n] = \{1, 2, 3, \ldots, n\}$

s.t.
- $V(S) = \sum_{i \in S} v_i$ is maximized
- $W(S) = \sum_{i \in S} w_i \leq W$

Brute Force
Similar to MWIS

$O(n2^n)$

ex: $W = 5$

	1	2	3	4
value	8	5	6	7
weight	2	1	3	4

What is optimal set?

A) $\{1, 3\}$

B) ~~$\{1, 1\}$~~ No repeats

C) ~~$\{1, 2, 3\}$~~ $W > 5$

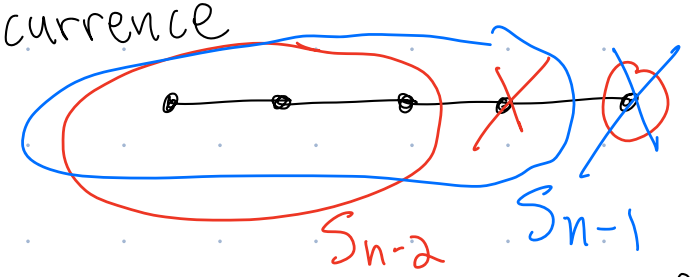
D $\{2, 3\}$ valid but low value

Dynamic Programming

1. Think about solution as sequence of choices. Final choice?

   MWIS: is each vertex in set or not → $v_n \in S$ or $v_n \notin S$

2. For each option, think about relevant subproblem + create recurrence

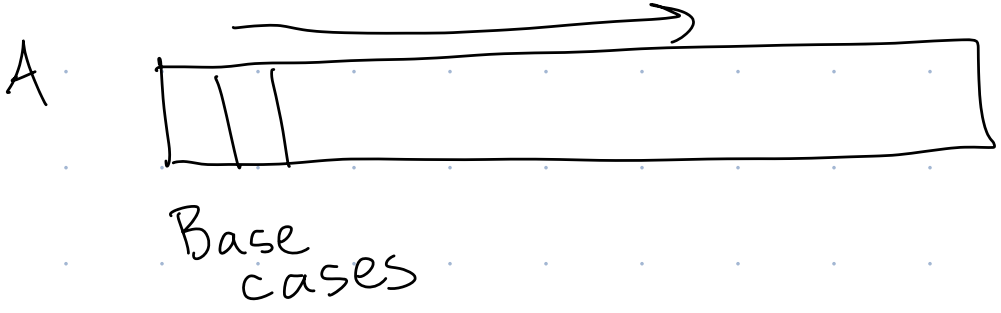


$$S_n = \begin{cases} S_{n-2} \cup \{v_n\} & \text{if } v_n \in S \\ S_{n-1} & \text{if } v_n \notin S \end{cases}$$

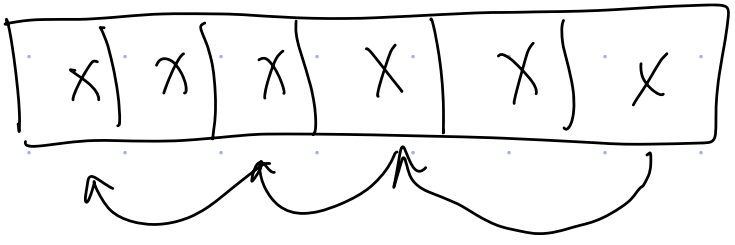
3. Convert into recurrence for objective function

$$W(S_n) \rightarrow A_n = \max\{A_{n-1}, A_{n-2} + w(v_n)\}, \text{ B.C.}$$

4. Pseudocode:



Work backwards



Knapsack Problem

Input: n items
- v_i is value of i^{th} item
- w_i is weight of i^{th} item

Capacity W (max weight)

Output: $S \subseteq [n]$ ← notation $\{1, 2, 3, \ldots, n\}$

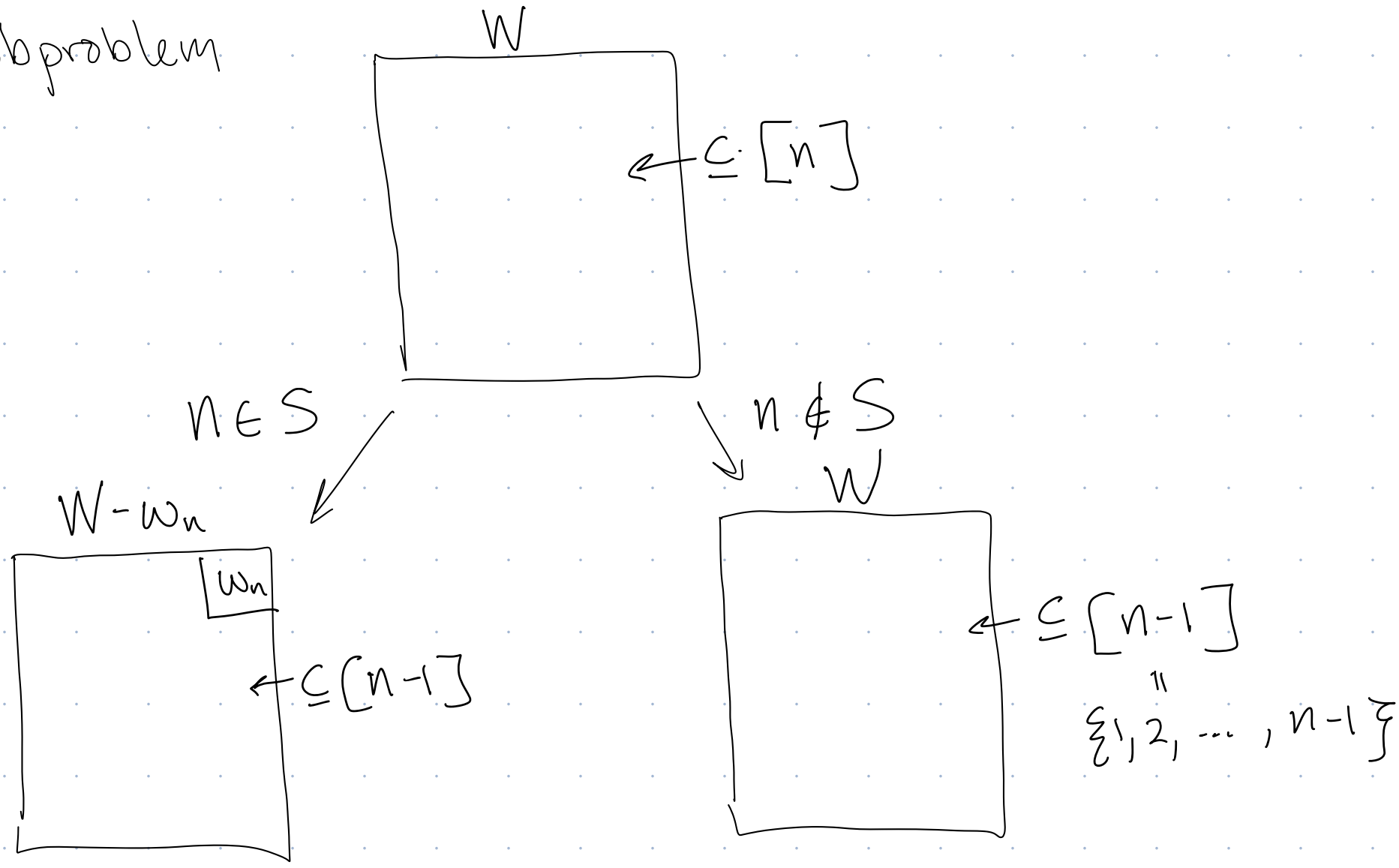
s.t. • $V(S) = \sum_{i \in S} v_i$ is maximized

• $W(S) = \sum_{i \in S} w_i \leq W$

1. Ideas for final option?

$n \in S$ or $n \notin S$

2a Subproblem



Define:

$$S_{i,C} = \begin{cases} S \subseteq [i] \\ \cdot\ W(S) \leq C \\ \cdot\ V(S) \text{ is maximized} \end{cases}$$

	1	2	3	4
value	6	5	8	7
weight	2	1	3	4

What is $S_{2,4}$?

$S_{2,4} \subseteq [2] = \{1, 2\}$

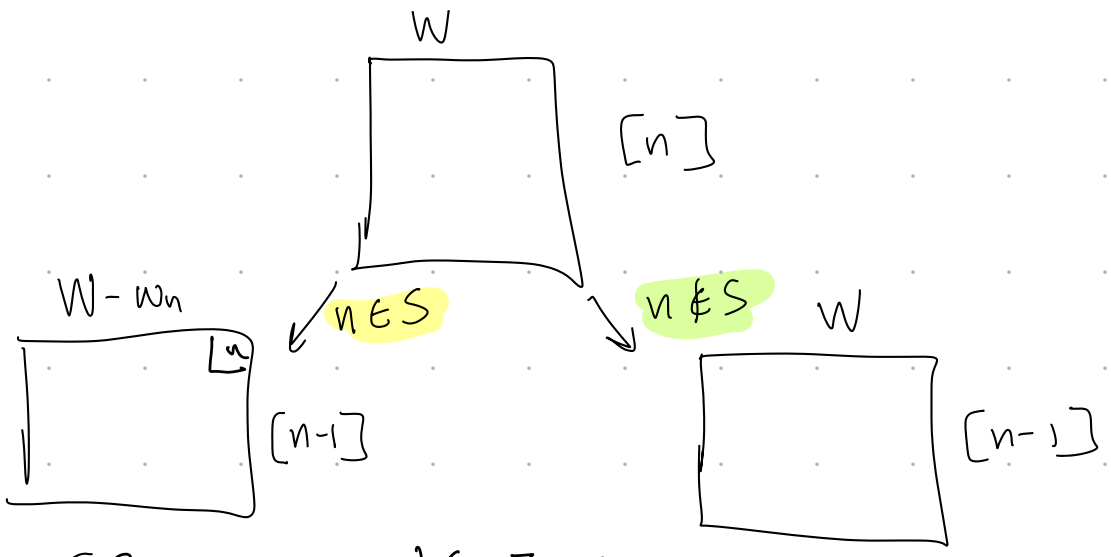
A) ~~$\{1, 1\}$~~ No repeats

B) $\{1, 2\}$ ✓

C) ~~$\{2, 3\}$~~

D) ~~$\{4\}$~~ $\notin [2]$

2b



$$S_{n,W} = \begin{cases} S_{n-1,\, W-w_n} \cup \{n\} & \text{if } n \in S \\ S_{n-1,\, W} & \text{if } n \notin S \end{cases}$$

$$V(S_{n,W}) = \begin{cases} \max\{ V(S_{n-1,\, W-w_n}) + v_n,\ V(S_{n-1,\, W}) \} \\ \text{Base cases} \end{cases}$$

$$V(S_{i,c}) = A[i,c]$$

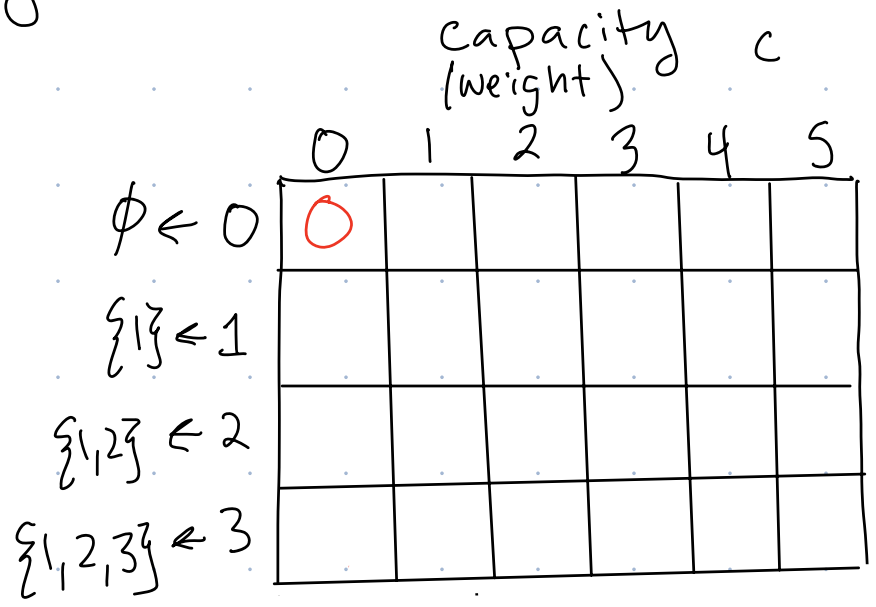
$$A[i,c] = \begin{cases} \max\{ A[i-1,\, C-w_i] + v_i,\ A[i-1,\, c] \} \\ \text{Base cases} \end{cases}$$

1. Fill out this array using recurrence for these items:

Input:

	1	2	3
value	6	5	8
weight	2	1	3

$W = 5$



$$A[i, c] = \begin{cases} \max\{A[i-1, c-w_i] + v_i, A[i-1, c]\} & \\ \quad i \in S \qquad\qquad i \notin S & \\ \boxed{} & \end{cases}$$

$A[0,0] = 0$

a) Determine Base Case(s)

b) Something else needed to properly use recurrence...

2. Write pseudocode to create A