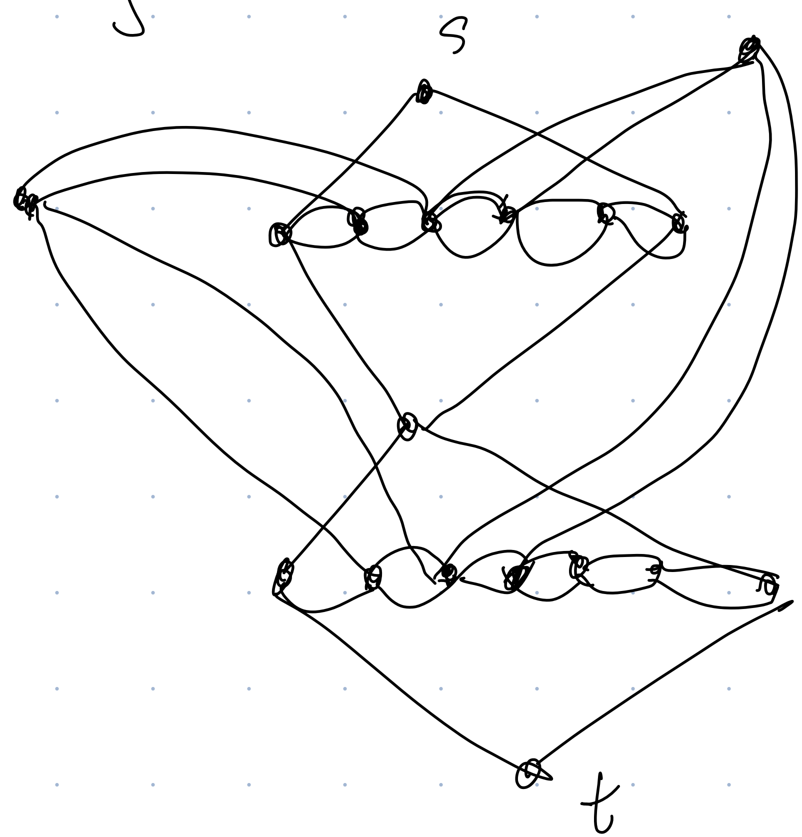
Learning Goals

- Describe Dijkstra's alg ✓
- Prove correctness of Dijkstra's alg.
- Analyze runtime of Dijkstra's alg



Exit Tickets

- Cheater
- Dijkstra Optimal?
- Why is rounding more accurate?

Shortest Path Problem

Input: $G = (V, E)$, $w : E \to \mathbb{R}^+$, $s, t \in V$, $|V| = n$, $|E| = m$, directed

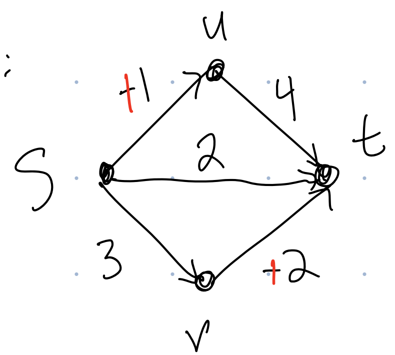
Output: Path P from s to t in G

e.g. $P = ((s,u), (u,v), \cdots (r,t))$ ← sequence of connected edges

such that $L(P) = \sum_{e \in P} w(e)$ is minimized

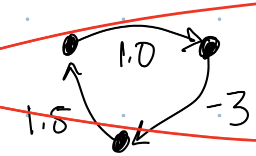
Example:

G:



Shortest Path: $\cancel{((s,v),(v,t))}$ $((s,t))$

~~Negative Cycle:~~



Not allowed

Various Approaches

- Greedy → Dijkstra's → edges have positive wt.
- Brute Force → Breadth-First Search → edges all have weight 1
- Dynamic → Bellman-Ford → edges can have negative wt
  → can work with global or distributed G
  → fails if must avoid neg. cycles

Dijkstra's Algorithm

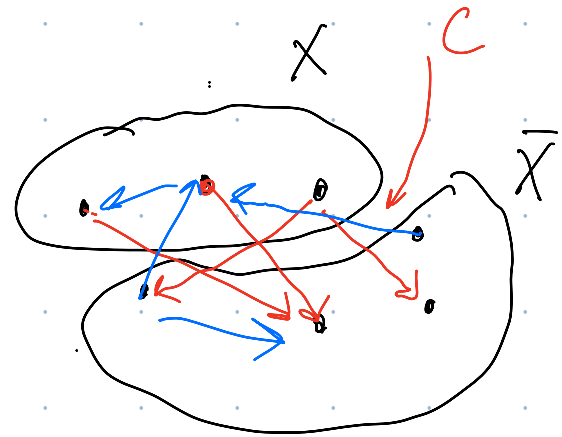
Input: $G = (V, E)$, $s \in V$, $|V| = n$, $w : E \to \mathbb{R}^+$

Output: n-dimensional arrays L, P s.t.

$L[v]$: length of shortest path from s to v in G

$P[v]$ = shorstest path from s to v in G

$X \leftarrow \{s\}$ // X is set of visited vertices



$L[s] \leftarrow 0$
$P[s] \leftarrow \phi$ } Base cases

While there is an edge from X to $\bar{X}$:

$C \leftarrow \{(u,v),\ u \in X,\ v \in \bar{X}\}$

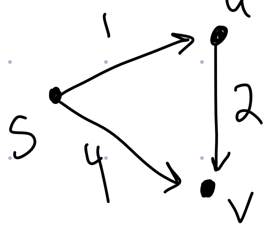
$(u^*, v^*) \leftarrow \underset{(u,v) \in C}{\operatorname{argmin}} \{L[u] + w(u,v)\}$ ← Dijkstra Criterion

$L[v^*] \leftarrow L[u^*] + w(u^*, v^*)$

$P[v^*] \leftarrow P[u^*] + (u^*, v^*)$

$X \leftarrow X \cup \{v^*\}$

Example



	L	P
s		

$X = \{s\}$

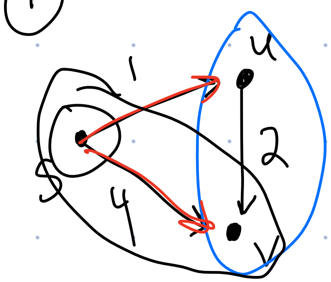
$L[s] = 0$

$P[s] = \phi$

While there is an edge from X to $\bar{X}$:

- $C \leftarrow \{(u,v) : u \in X, v \in \bar{X}\}$
- $(u^*, v^*) \leftarrow \underset{(u,v)\in C}{\text{argmin}} \{L[u] + w(u,v)\}$
- $L[v^*] \leftarrow L[u^*] + w(u^*, v^*)$
- $P[v^*] \leftarrow P[u^*] + (u^*, v^*)$
- $X \leftarrow X \cup \{v^*\}$

1



$X = \{s\}, \bar{X} = \{u, v\}$

$C = \{(s,u), (s,v)\}$

$L[s] + w(s,u)$ $L[s] + w(s,v)$

$0 + 1$ $0 + 4$

$u^* = s \quad v^* = u$

	L	P
s	0	ϕ
u	1	(s,u)

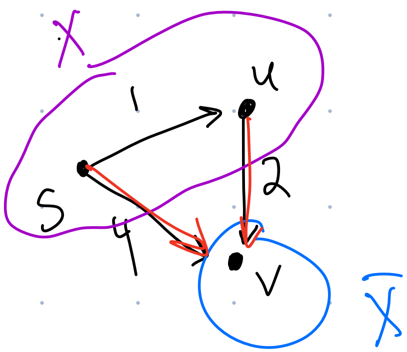
$X = \{s, v\}$

	L	P
s	0	ϕ
u	1	(s,u)
v	3	(s,u),(u,v)

2

$X = \{s, u\}$

$\bar{X} = \{v\}$



$C = \{(s,v), (u,v)\}$

$L[s] + w(s,v)$ $L[u] + w(u,v)$

$0 + 4$ $1 + 2$

$X = \{s, u, v\}$

Dijkstra's Algorithm

Input: $G=(V,E)$, $s \in V$, $|V|=n$, $w: E \to \mathbb{R}^+$

Output: n-dimensional arrays L, P s.t.

$L[v]$: length of shortest path from s to v in G

$P[v]$ = shorstest path from s to v in G

$X \leftarrow \{s\}$ // X is set of visited vertices

$$\left.\begin{array}{l} L[s] \leftarrow 0 \\ P[s] \leftarrow \phi \end{array}\right\} \text{Base case}$$

While there is an edge from X to $\bar{X}$:

$C \leftarrow \{(u,v) : u \in X, v \in \bar{X}\}$

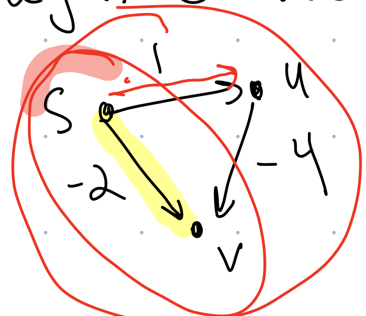
$(u^*, v^*) \leftarrow \underset{(u,v) \in C}{\operatorname{argmin}} \{L[u] + w(u,v)\}$

$L[v^*] \leftarrow L[u^*] + w(u^*, v^*)$

$P[v^*] \leftarrow P[u^*] + (u^*, v^*)$

$X \leftarrow X \cup \{v^*\}$

Show Dijkstra's alg fails with negative weights



Under what conditions can Dijkstra's alg have neg. weights but be successful?