Bellman-Ford Algorithm

Learning Goals

- Describe the Shortest Path problem + discuss applications
- Design a dynamic programming alg for shortest path

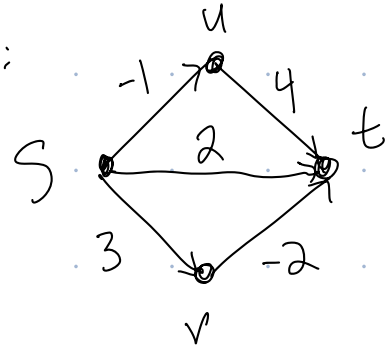
Shortest Path Problem

Input:

Output:

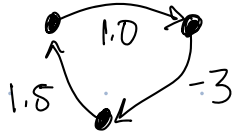
Example:

G:



Shortest Path:

Negative Cycle:



Various Approaches

- Greedy
- Brute Force
- Dynamic

Applications:

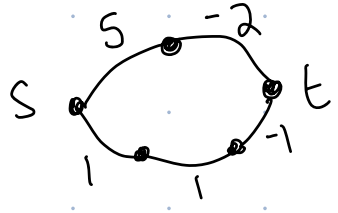
Cold Weather Plans?

Bartering	Stakeholders	Well-Being	Autonomy	Justice
Arbitrage				

Defining Subproblems

Clever idea:

What is $L(P_{t,1})$, $L(P_{t,2})$, $L(P_{t,3})$ for the following graph



A) $\infty, 3, 1$ B) $\infty, 2, 3$ C) $0, 2, 3$ D) $0, 3, 1$

Last choice a strategy makes?

Designing a Dynamic Prog. Alg.

- Recurrence relation for $P_{u,i}$ = shortest path from s to u that uses at most i edges

$P_{u,i} = \begin{cases} \_\_\_\_\_ & \text{if shortest path with } i \text{ edges goes through } v \text{ prior to } u \\ \_\_\_\_\_ & \text{if shortest path with } i \text{ edges goes through } w \text{ prior to } u \\ \vdots & \\ \_\_\_\_\_ & \text{if shortest path to } u \text{ uses less than } i \text{ edges} \end{cases}$

- Recurrence relation for objective function ($w(u,u) = 0$, $w(u,v) = \infty$ if no edge)

$L(P_{u,i}) = \begin{cases} \_\_\_\_\_ \\ \text{Base case: } \_\_\_\_\_ \end{cases}$

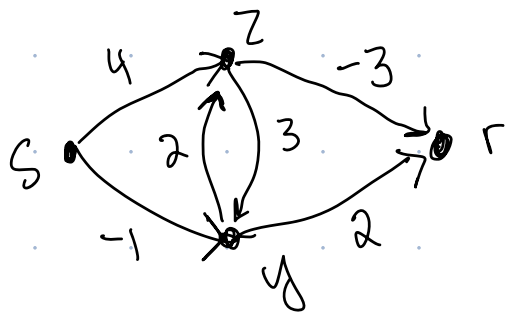
Helpful notation $\rightarrow \min_{u \in S} \{f(u)\}$

- Write Pseudocode to fill in A with $L(P_{u,i})$ values

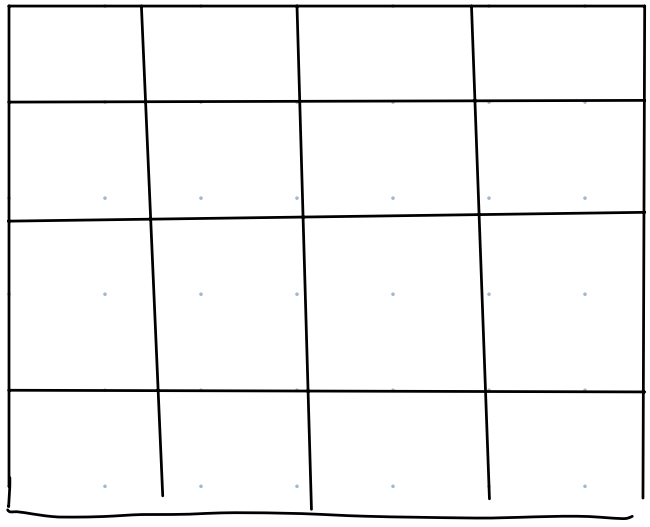
Input: $n \times n$ array w where $w[u,v]$ = weight of edge (u,v), $s, t \in [n]$.
$w[u,v] = \infty$ if no edge. $w[u,u] = 0 \ \forall \ u \in [n]$

Output: $? \times ?$ array A s.t. $A[u,i] = L(P_{u,i})$

Use your code to create A for the graph



A



What is the Runtime of Bellman-Ford?

A) $O(n)$ B) $O(n^2)$ C) $O(n^3)$ D) $O(nm)$ (n: # vertices, m: # edges)

Recall

Recurrence relation for objective function ($w(u,u) = 0$, $w(u,v) = \infty$ if no edge)

$$L(P_{u,i}) = \begin{cases} \_\_\_\_\_ \\ \text{Base case: } \_\_\_\_\_ \end{cases}$$

Reverse Adjacency List

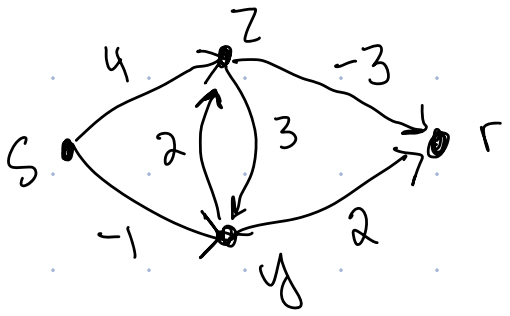
$W[s] = \emptyset$

$W[z] = [(s,4), (y,2)]$

$W[r] = [(z,-3), (y,2)]$

$W[y] = [(s,-1), (z,3)]$

e.g.



Initialize $A \leftarrow n \times n$ array filled with ∞'s.

$A[s,0] \leftarrow 0$

For $i \leftarrow 1$ to $n-1$:

    For $u \leftarrow 1$ to n:

        $A[u,i] = A[u,i-1]$

        For $v \leftarrow 1$ to n

            If $A[u,i] > A[v,i-1] + w[v,u]$

                $A[u,i] \leftarrow A[v,i-1] + w[v,u]$

What is the Runtime of Bellman-Ford with Reverse Adj Matrix?

A) $O(n)$ B) $O(n^2)$ C) $O(n^3)$ D) $O(nm + n^2)$

(n: # vertices, m: # edges)

Initialize $A \leftarrow n \times n$ array filled with ∞'s.

$A[s,0] \leftarrow 0$

For $i \leftarrow 1$ to $n-1$:

  For $u \leftarrow 1$ to n:

    $A[u,i] = A[u,i-1]$

    For $v : (v,u) \in E$:

      If $A[u,i] > A[v,i-1] + w[v,u]$

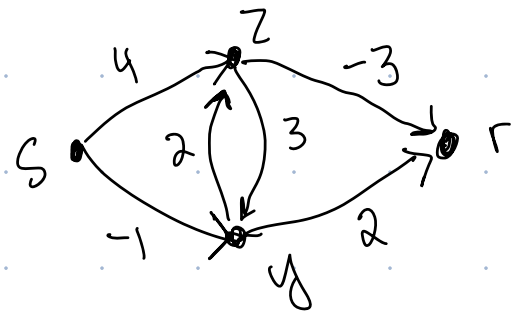
        $A[u,i] \leftarrow A[v,i-1] + w[v,u]$

$W[s] = \phi$

$W[z] = [(s, 4), (y, 2)]$

$W[r] = [(z, -3), (y, 2)]$

$W[y] = [(s, -1), (z, 3)]$



Distributed Bellman-Ford