CS333 - Problem Set 7

1. If $\{|\psi_{00}\rangle, |\psi_{01}\rangle, |\psi_{10}\rangle, |\psi_{11}\rangle\}$ is an orthonormal basis for two qubits, then show that there is a unitary gate that acts as:
$$|\psi_i\rangle \to |i\rangle \tag{1}$$
where $|i\rangle$ is the standard basis state. The following is very very useful to know:
$$\sum_i |i\rangle\langle i| = I \tag{2}$$
where the sum is over all standard basis states $|i\rangle$.

2. In class, we discussed how classical probabilistic computation can be represented mathematically using vectors to represent the probabilistic state of the system, and left stochastic matrices to represent operations. In this problem, we'll examine a similar approach for reversible deterministic (non-probabilisitc) classical computation. If the system consists of n bits, there are 2^n possible states of the system. We can represent the state $s \in \{0,1\}^n$ using the vector $|s\rangle$ that is length 2^n and contains all 0s except for a 1 in the sth position, where we count in binary, and start with 0. For example, if we have a 3-bit system in the state (010) (first and third bits are 0, and the second bit is 1) we would represent the system with the vector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3}$$

since in converting 010 to decimal we get 2, and we are counting starting at 0, so we get a 1 in the 3rd element of the vector.

(Here $|s\rangle$ doesn't represent a quantum state, but rather the standard basis vector, which we are intepreting now as a state of a classical, deterministic, reversible computer.)

(a) Please give a matrix representation of each of the following reversible, deterministic gates:
    i. NOT (acts on a single bit and flips the value)
    ii. TOFFOLI (acts on 3 bits and flips the value of the third bit if both of the first bits have value 1)

(b) What class of matrices describe the set of possible gates in this model? (In the case of quantum, we had *unitary*, and in the case of probabilistic, we had *left stochastic*, so what is the corresponding description for deterministic reversible?)

(c) NOT and TOFFOLI are universal for this model of computation, which means that any deterministic (non-random) classical computation can be implemented using just these two gates. Are the matrices from part (b) also valid left stochastic matrices? Are these matrices also valid unitaries? Based on your answers, please comment on the ability of probabilistic computers to simulate deterministic computers, and the ability of quantum computers to simulate determistic computers. If it is possible to do this simulation, briefly explain how you would do it.

3. Let $f : \{0,1\}^2 \to \{0,1\}$ be a black-box function whose input is two bits, where f takes the value 1 on exactly one input value (and is zero on the other three input values). For example we could have $f(00) = 0$, $f(01) = 1$, $f(10) = 0$ and $f(11) = 0$. We would like to create an algorithm that identifies which input to f gives output 1. We call this input the "marked item.' 'So in our example, the algorithm should output 01, and we say 01 is the "marked item". This is a small example of a "search problem."

(a) Fill in the following truth table with the 4 possible black box functions: The possible functions are $f_{00}, f_{01}, f_{10}, f_{11}$ where the subscript indicates which item outputs 1.
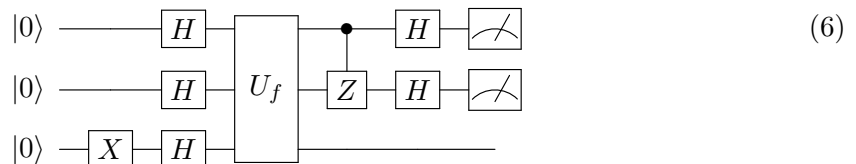
x_1	x_2	$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$

(4)

(b) How many classical queries to f are needed to solve one-out-of-four search in the worst case?

(c) Let $|x_1\rangle$, $|x_2\rangle$, and $|y\rangle$ be single qubit standard basis states. Then the quantum version of f is the unitary U_f that acts on standard basis states as:

$$U_f|x_1, x_2\rangle|y\rangle = |x_1, x_2\rangle|y \oplus f(x_1, x_2)\rangle,$$

(5)

where $\oplus$ is XOR, that is, addition mod 2. Now consider the following circuit:

(6)

i. What is the state of the system immediately before the unitary U_f?

ii. What is the state of the system immediately after the unitary U_f? Show that we get a phase kickback like in Deutsch's algorithm

iii. Show that if we consider the 4 possible states immediately after U_f acts (one for each of the functions f_{00}, f_{01}, f_{10}, f_{11}), the possible states form an orthonormal basis.

2

iv. The next gate is Control-Z. We describe its its action on the standard basis: if the first qubit is in the $|1\rangle$ state, it applies the gate Z to the second qubit, and if the first qubit is in the state $|0\rangle$, it does nothing (applies the identity). For each of the functions f_{00}, f_{01}, f_{10}, f_{11}, analyze the outcome of the circuit, and describe how the measurement result provides the location of the marked item.

v. Describe at a high level (using words like phase kickback and superposition and using the result from Problem 1 on this Pset) how this quantum algorithm works.