S. KIMMEL

Quantum Operations (gates)

Reversible transformations

Mathematical representation: Unitary matrix $U \in \mathbb{C}^{2\times 2}$ ⇐ 2×2 complex matrix

(qubit operation)

$$\Updownarrow$$

$$U \underbrace{U^{+}}_{\text{conj. transpose}} = U^{+}U = \underbrace{I}_{\text{identity } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

In action

$$\underbrace{|\psi\rangle}_{\text{vector}} \xrightarrow{U \text{ acts}} \underbrace{U|\psi\rangle}_{\substack{\text{matrix}\cdot\text{vector} \\ =\text{vector}}}$$

matrix multiplication

Q: What unitary represents

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \rightarrow \left(\tfrac{\pi}{4}\right) \rightarrow \begin{pmatrix} \cos(\theta + \frac{\pi}{4}) \\ \sin(\theta + \frac{\pi}{4}) \end{pmatrix}$$

π/4 rotation lens

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Q: Show if U is a unitary and $|\psi\rangle$ is a state, $U|\psi\rangle$ is a state.

Q: Why are unitary operations invertible?

Pf: We want that $|\psi'\rangle = U|\psi\rangle$ is normalized,

but $|\psi'\rangle$ is normalized $\iff \langle\psi'|\psi'\rangle = 1$,

and $\langle\psi'| = (U|\psi\rangle)^\dagger = (|\psi\rangle)^\dagger U^\dagger = \langle\psi|U^\dagger$

so $\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi| I |\psi\rangle = \langle\psi|\psi\rangle = 1$

- If U is a unitary, U^{-1} (inverse of U) is a unitary. Consequently quantum operations are reversible!

Pf By def, $UU^\dagger = \mathbb{I}$, so $U^{-1} = U^\dagger$.

Now $U^\dagger$ is a unitary if $U^\dagger (U^\dagger)^\dagger = \mathbb{I}$.

But $(U^\dagger)^\dagger = U$, so $U^\dagger (U^\dagger)^\dagger = U^\dagger U = \mathbb{I}$

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- Important single qubit unitaries:

Paulis: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hadamard: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Ket-bra

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 & 1\cdot 0 \\ 0\cdot 1 & 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)|+\rangle = |0\rangle\underbrace{\langle 0|+\rangle}_{\frac{1}{\sqrt{2}}} - |1\rangle\underbrace{\langle 1|+\rangle}_{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big) = |-\rangle$$

$$\big(|\psi\rangle\langle\phi|\big)^\dagger = |\phi\rangle\langle\psi|$$