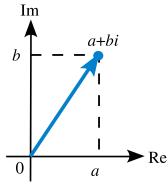# CS333 - Pre Class Math

1 Complex Numbers

A complex number is written as $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. It is useful to think about complex numbers as points on a plane:

If $x = a + bi$, then the *complex conjugate* of x, denoted x^*, is $a - bi$. If w and x are complex numbers, then

$$(wx)^* = w^* x^*$$
$$(w + x)^* = w^* + x^* \tag{1}$$

Question: If we think of complex numbers as points on a plane, and conjugation as a function that takes a complex number to its conjugate, how does the operation of conjugation transform points on a plane?

For $w \in \mathbb{R}$, Euler's formula is the following:

$$e^{iw} = \cos(w) + i\sin(w). \tag{2}$$

This formula lets us use a complex exponential to represent a complex numbers.

Question: If $w \in \mathbb{R}$, which of the following is the complex conjugate of e^{iw}?

$$e^{-iw}, \qquad \cos(w) - i\sin(w), \qquad -\cos(w) + i\sin(w), \qquad e^{w} \tag{3}$$

If $x = a + bi$, we denote the magnitude of x as $|x|$, where $|x| = \sqrt{xx^*}$.

Question: Explain why $|e^{ix}| = 1$ twice, using each side of Euler's formula and the definition of magnitude. (For the left hand side, you'll also need properties of products of exponentials, and for the right hand side, you'll also need properties of trigonometric functions.)

Question: What is a simpler expression for $e^{i3\pi/2}$, and where does it appear as a point on the plane?

---

2 Vector Spaces

We will deal with vector spaces $\mathbb{C}^d$. $\mathbb{C}^d$ is the set of column vectors of length d (dimension d) whose elements are complex numbers. So for example

$$\begin{pmatrix} 1 \\ i \\ e^{2i} \end{pmatrix} \in \mathbb{C}^3. \tag{4}$$

If $\mathbf{x} \in \mathbb{C}^d$, then the conjugate transpose of $\mathbf{x}$, denoted by $\mathbf{x}^\dagger$, is the d-dimensional row vector where the jth element of $\mathbf{x}^\dagger$ is the complex conjugate of the jth element of $\mathbf{x}$. For example, if

$$\mathbf{x} = \begin{pmatrix} 1 \\ i \\ e^{2i} \end{pmatrix} \qquad \text{then} \qquad \mathbf{x}^\dagger = \begin{pmatrix} 1, & -i, & e^{-2i} \end{pmatrix}. \tag{5}$$

More generally, if you have a matrix $\mathbf{A}$, then $\mathbf{A}^\dagger$ denotes the conjugate transpose of $\mathbf{A}$, where you take the transpose of the matrix, and then take the complex conjugate of each element. For example

$$\begin{pmatrix} 1, & 2, & 3 \\ -1i, & -2i, & -3i \end{pmatrix}^\dagger = \begin{pmatrix} 1, & 1i \\ 2, & 2i \\ 3, & 3i \end{pmatrix}$$

Given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^d$, we can take the inner product $\mathbf{y}^\dagger \mathbf{x}$ by doing matrix multiplication between $\mathbf{y}^\dagger$ and $\mathbf{x}$. That is: $\mathbf{y}^\dagger \mathbf{x} = \sum_{i=1}^{d} y_i^* x_i$, where x_i is the ith element of $\mathbf{x}$ and y_i is the

ith element of $\mathbf{y}$. For example, if $\mathbf{x}$ is as above, and $\mathbf{y} = \begin{pmatrix} i \\ 1+i \\ 2 \end{pmatrix}$, then

$$\mathbf{y}^\dagger \mathbf{x} = \begin{pmatrix} -i, & 1-i, & 2 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ e^{2i} \end{pmatrix} = -i \times 1 + (1-i) \times i + 2 \times e^{2i} = -i + i + 1 + 2e^{2i}$$

$$= 1 + e^{2i}. \tag{6}$$

Question: If $\mathbf{x} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, what is $\mathbf{y}^\dagger \mathbf{x}$? What is $\mathbf{x}^\dagger \mathbf{y}$?

Question: If $\mathbf{x}, \mathbf{y}$ are any vectors in $\mathbb{C}^d$, explain why $\left(\mathbf{y}^\dagger \mathbf{x}\right)^* = \mathbf{x}^\dagger \mathbf{y}$.

Question: Show that the inner product follows the distributive property. That is, if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^d$, explain why $\mathbf{z}^\dagger (\mathbf{x} + \mathbf{y}) = \mathbf{z}^\dagger \mathbf{x} + \mathbf{z}^\dagger \mathbf{y}$.

A *basis* for a vector space $\mathbb{C}^d$ is a set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_d\}$ such that for every vector $\mathbf{x} \in \mathbb{C}^d$, there is a unique set of complex numbers $\{a_1, \ldots, a_d\}$ such that

$$\mathbf{x} = \sum_{j=1}^d a_j \mathbf{v}_j. \tag{7}$$

If you have a set of d vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_d\}$ each in $\mathbb{C}^d$ such that

$$\mathbf{v}_j^\dagger \mathbf{v}_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases} \tag{8}$$

then they form a basis for $\mathbb{C}^d$. We call such a basis an *orthonormal basis*.

For example you can verify that:

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{9}$$

from an orthonormal basis for $\mathbb{C}^2$ because there are two of them, and they satisfy Eq. (8). In quantum computing, we will exclusively deal with orthonormal bases.

Given a vector $\mathbf{x} \in \mathbb{C}^d$, it is often helpful to write that vector in terms of a given orthonormal basis $\{\mathbf{v}_1, \ldots \mathbf{v}_d\}$. In other words, we would like to find the complex numbers

a_j as in Eq. (7). When $\{\mathbf{v}_1, \ldots \mathbf{v}_d\}$ is an orthonormal basis, finding this decomposition is fairly straightforward. We can apply $\mathbf{v}_i^\dagger$ to both sides of equation Eq. (7):

$$
\begin{aligned}
\mathbf{v}_i^\dagger \mathbf{x} = \mathbf{v}_i^\dagger \left(\sum_{j=1}^{d} a_j \mathbf{v}_j \right) \\
= \sum_{j=1}^{d} a_j \mathbf{v}_i^\dagger \mathbf{v}_j \\
= a_i \qquad\qquad (10)
\end{aligned}
$$

where the second line comes from the distributive property, and the final line comes from Eq. (8).

Question: Consider the orthonormal basis $\{\mathbf{v_1}, \mathbf{v_2}\}$ for $\mathbb{C}^2$ where $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Write $\mathbf{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ in this basis. (In other words, write $\mathbf{x} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2$ for some complex numbers a_1 and a_2.)

Let $\mathbb{C}^{n \times m}$ denote the set of matrices with n rows and m columns and complex elements. Let $\mathbf{A} \in \mathbb{C}^{n \times m}$ and $\mathbf{B} \in \mathbb{C}^{p \times q}$. Then the *tensor product* (technically the Kronecker product) of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$. If the element in the ith row and jth column of $\mathbf{A}$ is A_{ij}, then

$$
\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_{11}\mathbf{B}, & A_{12}\mathbf{B}, & \ldots & A_{1m}\mathbf{B} \\ A_{21}\mathbf{B}, & A_{22}\mathbf{B}, & \ldots & A_{2m}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}\mathbf{B}, & A_{n2}\mathbf{B}, & \ldots & A_{nm}\mathbf{B} \end{pmatrix}.
$$

For example,

$$
\begin{pmatrix} 1, & 2, & 3 \\ -1, & -2, & -3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & 2\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & 3\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \\ -1\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & -2\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, & -3\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1i & 2i & 3i \\ 0 & 0 & 0 \\ -1 & -2 & -3 \\ -1i & -2i & -3i \end{pmatrix}.
$$

Question: If $\mathbf{A}$ is an $n \times m$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, what are the dimensions of $\mathbf{A} \otimes \mathbf{B}$?

The tensor product has the following properties:

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} \tag{11}$$

$$(\mathbf{B} + \mathbf{C}) \otimes \mathbf{A} = \mathbf{B} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{A} \tag{12}$$

$$(\mathbf{A} \otimes \mathbf{B})^\dagger = \mathbf{A}^\dagger \otimes \mathbf{B}^\dagger \qquad \text{Note! the order stays the same!} \tag{13}$$

$$(\mathbf{A} \otimes \mathbf{B}) \cdot (\mathbf{C} \otimes \mathbf{D}) = \mathbf{A} \cdot \mathbf{C} \otimes \mathbf{B} \cdot \mathbf{D}, \tag{14}$$

where $\cdot$ denotes regular matrix multiplication.

Question: Consider the orthonormal basis $\{\mathbf{v_1}, \mathbf{v_2}\}$ for $\mathbb{C}^2$ where $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Show that $\{\mathbf{v_1} \otimes \mathbf{v_1}, \mathbf{v_1} \otimes \mathbf{v_2}, \mathbf{v_2} \otimes \mathbf{v_1}, \mathbf{v_2} \otimes \mathbf{v_2}\}$ is an orthonormal basis for $\mathbb{C}^4$.

Question: In this problem, we'll show that the previous problem generalizes. Suppose we have an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ for $\mathbb{C}^d$, and an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_f\}$ for $\mathbb{C}^f$. Show that the set of vectors consisting of all possible pairs of tensor products $\mathbf{v}_i \otimes \mathbf{u}_j$ form an orthonormal basis, and say which space they form a basis for. (Use the tensor product properties!)