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Basic Algorithm:

1. Prepare $|0\rangle_A |0\rangle_B$ (A: N-dim, B: W-dim)

2. Apply QFT_N to A

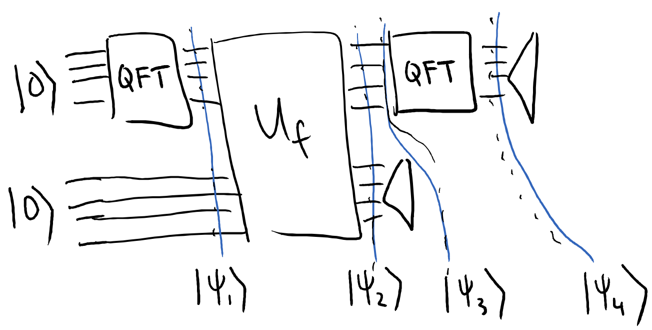
3. Apply U_f to A, B

4. Measure B in standard basis

5. Apply QFT_N to A

6. Measure A in standard basis

Q: Write as circuit -



Full Algorithm

1. Run basic algorithm twice. Get outcomes y, y'. Do Classical postprocessing on y, y'. Outcome of postprocessing is r with high probability. Check by querying $f(1)$ and $f(r+1)$

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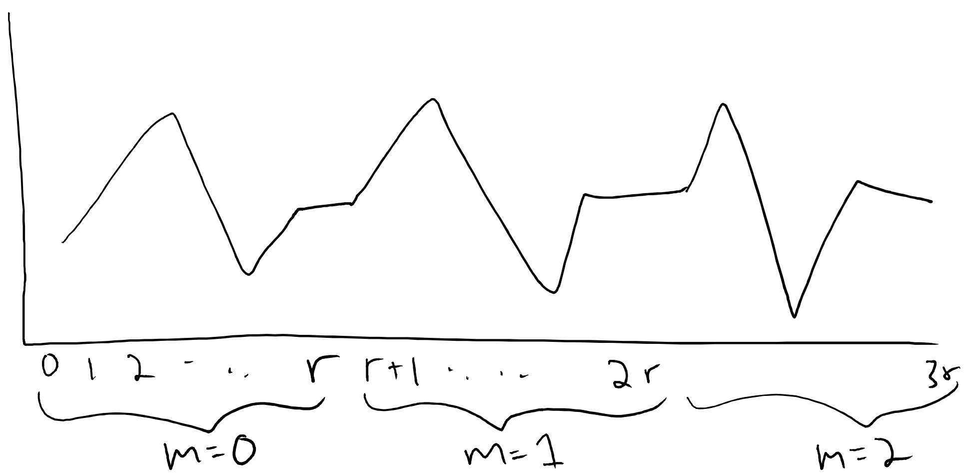
1. $$|\psi_1\rangle = \left(QFT\,|0\rangle\right)|0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle_A |0\rangle_B$$

2. $$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} U_f |x\rangle|0\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} |x\rangle|f(x)\rangle$$

Recall: $f(x)$ is periodic. Let's write $x = m\underset{\text{period}}{r} + b$

Q: What is $f(mr+b)$ equal to?

A) $f(r)$ B) $f(m)$ $\boxed{\text{C) } f(b)}$ D) $f(mr)$



$b \in [r]$

$m \in \left[\frac{N}{r}\right]$

$m=i$, $b=j$ corresponds to j^{th} element of i^{th} block of r

Rewrite x as $x = mr+b$. $\sum_x$ becomes $\sum_m \sum_b$

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$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle|f(x)\rangle$$

$\Downarrow$ with change of variables

$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{b=0}^{r-1} \sum_{m=0}^{m_b-1} |mr+b\rangle_A |f(mr+b)\rangle$$

m_b is # blocks where b occurs. If r does not divide N evenly, some values of b will not occur in last block

$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{b=0}^{r-1} \sum_{m=0}^{m_b-1} |mr+b\rangle|f(b)\rangle$$

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3. Measure B register in standard basis.

Use partial measurement to analyze:

$$|\psi_2\rangle = \sum_{b=0}^{r-1}\left(\frac{1}{\sqrt{N}}\sum_{m=0}^{m_b-1}|mr+b\rangle\right)_A |f(b)\rangle_B$$

standard basis states, different for each b by assumption that values are unique within a period

$$|\psi_2\rangle = \sum_{b=0}^{r-1}\underbrace{\frac{1}{\alpha\sqrt{N}}}_{\text{Related to probability of outcome}}\underbrace{\left(\alpha\sum_{m=0}^{m_b-1}|mr+b\rangle\right)_A}_{\text{Want this to be normalized}}|f(b)\rangle_B$$

Q. What is the (approximate) value of α?

A) $\frac{1}{\sqrt{N}}$ B) $\frac{1}{\sqrt{b}}$ C) $\frac{1}{\sqrt{m}}$ $\boxed{\text{D) } \sqrt{\frac{r}{N}}}$

Because $m_b = \frac{N}{r}$ or $\frac{N}{r} - 1$

Suppose we get outcome $|s\rangle$. Let b^* be value such that $f(b^*) = s$. Then after measurement, state collapses to:

$$|\psi_3\rangle = \left(\frac{1}{\sqrt{m_{b^*}}}\sum_{m=0}^{m_{b^*}-1}|mr+b^*\rangle\right)_A |f(b^*)\rangle_B$$

We never do anything else with B system. Since partial measurement leaves us in tensor state of A & B, we can ignore B from here on.

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4. Now apply QFT_N to A:

Distribute!

$$|\psi_4\rangle = QFT_N \frac{1}{\sqrt{m_{b^*}}} \sum_{m=0}^{m_{b^*}-1} |mr+b^*\rangle = \frac{1}{\sqrt{m_{b^*}}} \sum_{m=0}^{m_{b^*}-1} QFT_N|mr+b^*\rangle$$

$$= \frac{1}{\sqrt{m_{b^*}}} \sum_{m=0}^{m_{b^*}-1} \left(\frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i \frac{(mr+b^*)y}{N}} |y\rangle \right)$$

$$= \frac{1}{\sqrt{N m_{b^*}}} \sum_{m=0}^{m_{b^*}-1} \left(\sum_{y=0}^{N-1} e^{\frac{2\pi i m r y}{N}} e^{\frac{2\pi i b^* y}{N}} |y\rangle \right)$$

Switch order of summation

$$= \frac{1}{\sqrt{N m_{b^*}}} \sum_{y=0}^{N-1} \left(\sum_{m=0}^{m_{b^*}-1} e^{\frac{2\pi i b^* y}{N}} e^{\frac{2\pi i m r y}{N}} |y\rangle \right)$$

Factor out $e^{2\pi i b^* y/N}$

$$= \frac{1}{\sqrt{N m_{b^*}}} \sum_{y=0}^{N-1} e^{\frac{2\pi i b^* y}{N}} \left(\sum_{m=0}^{m_{b^*}-1} e^{\frac{2\pi i m r y}{N}} \right) |y\rangle$$

$$|\psi_4\rangle = \sum_{y=0}^{N-1} \frac{1}{\sqrt{N m_{b^*}}} e^{2\pi i b^* y/N} \left(\sum_{m=0}^{m_{b^*}-1} e^{2\pi i m r y/N} \right) |y\rangle$$

5. Measure in standard basis:

$$\Pr(\text{outcome } |y\rangle) = \left| \frac{1}{\sqrt{N m_{b^*}}} e^{2\pi i b^* y/N} \cdot \sum_{m=0}^{m_{b^*}-1} e^{2\pi i m r y/N} \right|^2$$

(*)

$$= \left| \frac{1}{\sqrt{N m_{b^*}}} e^{2\pi i b^* y/N} \right|^2 \left| \boxed{\sum_{m=0}^{m_{b^*}} e^{-2\pi i m r y/N}} \right|^2$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{1}{N m_{b^*}} \approx \frac{1}{r} \qquad\qquad ?$$

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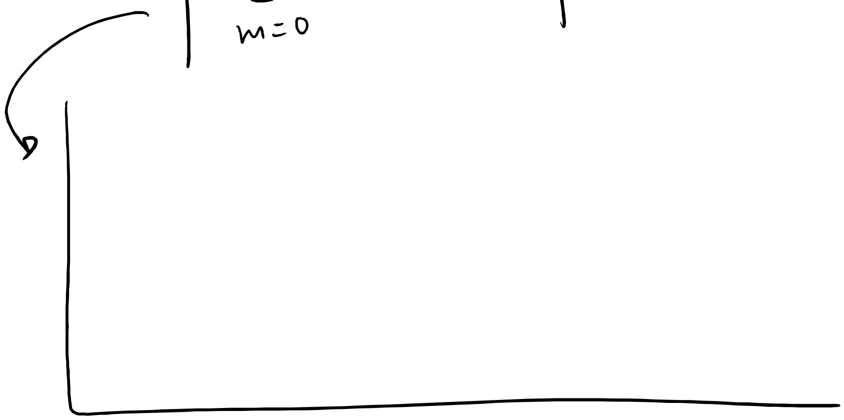
Q: Plot $\left| \sum_{m=0}^{m_{b^*}} e^{-2\pi i m r y / N} \right|^2$ as a function of y

Probability of outcome $|y\rangle$

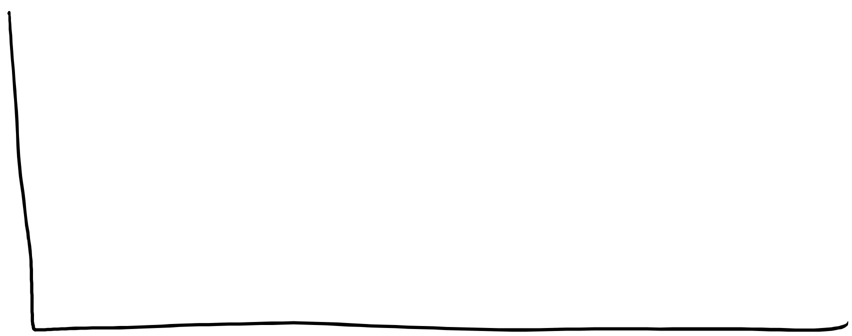


y

Q: Before QFT, we had

$$|\psi_3\rangle = \frac{1}{\sqrt{m_{b^*}}} \sum_{m=0}^{m_{b^*}-1} |mr + b^*\rangle$$

Why not measure $|\psi_3\rangle$? Plot probability of outcome $|y\rangle$



$|y\rangle$