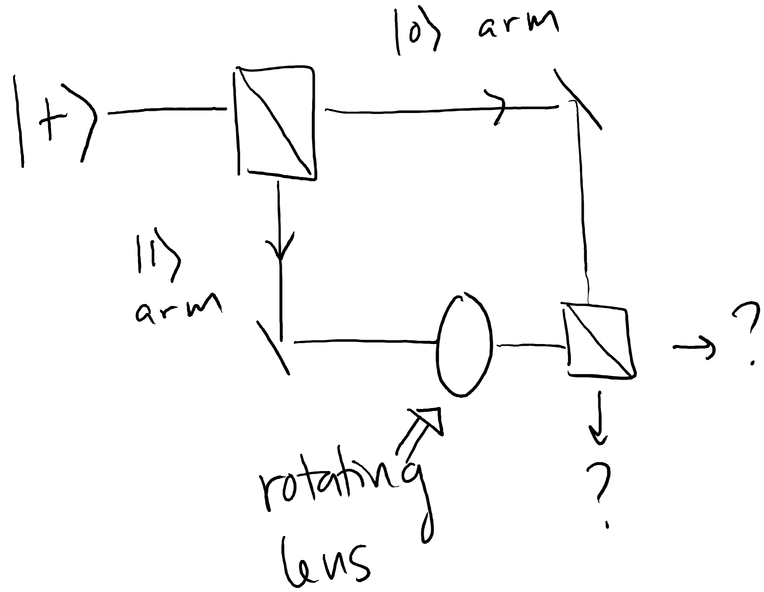
CS333 - Problem Set 4



1.

Consider the interferometer in the figure. In each of the following cases, what will be the polarization of the photon that exits the interferometer, and will it exit downwards or rightwards (or both?) Note in both cases the lens is acting as a gate, NOT a measurement. The lens only affects the polarization, not the position of the photon, and only affects the part of the superposition in the lower arm of the interferometer.

(a) The lens acts on the polarization of standard basis states in the following way:

$$\begin{aligned} |0\rangle &\rightarrow |1\rangle \\ |1\rangle &\rightarrow -|0\rangle. \end{aligned} \tag{1}$$

(So for example, the lens causes $|+\rangle$ to shift to $-|-\rangle$.)

(b) The lens acts on the polarization of standard basis states in the following way:

$$\begin{aligned} |0\rangle &\rightarrow -|0\rangle \\ |1\rangle &\rightarrow -|1\rangle. \end{aligned} \tag{2}$$

(In other words, this lens add a -1 "phase" to the photon.)

(c) In Problem Set 2, no. 6, you showed that if you multiply a state by -1, it is not physically different from a state that is not multiplied by -1. (Note that $-1 = e^{i\pi}$.) However, in part (b) of this problem, we see that multiplying by -1 does change the physical outcome relative to the case with no lens in the interferometer. (What happens with no lens?) What is different here from PS2 no. 6?

2. The Bloch sphere is a useful tool for visualizing single qubit states, gates, and measurements. In the next few questions, we'll investigate this tool.

The most general way we can write a 2 qubit state (that fulfills the normalization condition), is

$$\begin{pmatrix} e^{i\chi}\cos\theta \\ e^{i\xi}\sin\theta \end{pmatrix} \tag{3}$$

However, in PS2 no. 6, we saw that we can multiply both parts of a quantum state by a "phase" $e^{i\omega}$ and that has no physical effect on the state. (A phase is what we call a complex number whose absolute value squared is 1.) So we can use this phase freedom to always choose the $|0\rangle$ amplitude of the state to be real. Thus we can always represent a single qubit state in the following way, using only two parameters: θ and φ :

$$|\psi(\theta,\varphi)\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix} \tag{4}$$

where $\theta \in [0, \pi/2]$ and $\varphi \in [0, 2\pi)$.

Next, note that we can label every single point on the surface of a sphere at radius 1 from the origin using two parameters: θ and φ, where $\theta \in [0, \pi]$ (note π, not $\pi/2$ in this case) and $\varphi \in [0, 2\pi)$. We do this by using θ to represent the polar angle (the angle between the point in question and the north pole, kind of like latitude) and using φ to represent the azimuthal angle (longitude). (If you would like more info on this, please look up spherical coordinates. See also Fig. 1.)

Because of this similarity in parameters between the qubit and the sphere, you can associate each qubit state with a unique point on the surface of a sphere. To do this, we identify the state $|\psi(\theta/2,\varphi)\rangle$ with the point on the sphere with polar angle θ and azimuthal angle φ, as in the following diagram. (**Important:** the θ in Eq. (4) is not the same as the θ in Fig. 1; they differ by a factor of 2, so be careful of this when moving from one representation to another. However, φ's are the same in the equation and Fig. 1.) When we are thinking of the sphere as a space where qubit states live, we call it the Bloch Sphere.

As practice, first verify that the vector **z** (north pole direction) corresponds to $|0\rangle$, and **-z** (south pole direction) corresponds to $|1\rangle$.

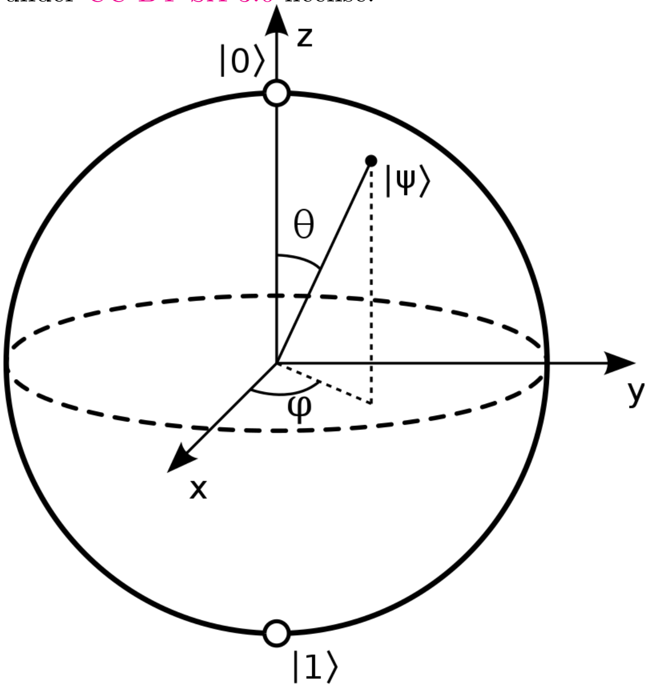
(a) What qubit states do the vectors **x**, **y**, **-x**, and **-y** correspond to? (See figure. Note **y** is 90° from **x** on the equator.)

(b) What is the absolute value squared of the inner product of any two states that are at 90° from each other? 180° from each other? (You do not need to prove this result. Instead extrapolate from trying examples.)

(c) Using part (b), what does a single qubit measurement correspond to on the Bloch Sphere? (Again you do not need to prove this, just extrapolate from examples.)

(d) Single qubit unitaries correspond to rotations of the sphere. (This intuitively makes sense because unitaries transform one state to another, and a rotation transforms one

Figure 1: Figure courtesy Smite-Meister from Wikipedia "Bloch sphere"; used without changes under CC BY-SA 3.0 license.



vector to another.) A rotation is defined by two quantities: an axis of rotation, and an angle of rotation. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \tag{5}$$

is a rotation around the **z** axis (because this unitary doesn't change the state $|0\rangle$). The direction of rotation is found using the "right hand rule:" point your right thumb in the direction of the rotation (in this case **z**), and then the rotation is in the direction your fingers move when you start with an open hand and then close it. So in this case, to figure out the angle of rotation, we can test what happens to states on the effective equator of the rotation axis. In this case, because the rotation is about the north pole, the effective equator is the actual equator, and we can look at what the unitary does to a state on the equator. For example, $|+\rangle$ is on the equator, and we see that the unitary turns $|+\rangle$ into $|\leftarrow\rangle$. This is a $90°$ rotation, but in the opposite direction of the right hand rule, so this unitary corresponds to a rotation of $-\pi/2$ $(-90°)$ of the Bloch sphere. (Or you could also think of this as a $3\pi/2$ rotation.) Alternatively, we could describe this unitary as a rotation around the **-z** axis by an angle of $\pi/2$. (There are always two possible choices for the axis of rotation: a vector and it's negation, however you need to make sure that the angle you choose is in the proper direction relative to the right hand rule.)

What is the axis of rotation and angle of rotation of the Bloch sphere corresponding to each of the following unitaries?

i. I

ii. X

iii. Z

iv. Y

v. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

vi. H

(e) Based on the examples in part (d), please extrapolate the connection between the eigenvectors and eigenvalues of a single qubit unitary, and the corresponding Bloch sphere rotation.

3. In class, we discussed a rotating lens that corresponds to a unitary U_χ that acts in the following way: if a photon with polarization $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ enters the lens, for any value of θ, the lens transforms the state, producing a photon with polarization $\cos(\theta+\chi)|0\rangle + \sin(\theta+\chi)|1\rangle$. Please describe the unitary operation U_χ in matrix form and show that your matrix is in fact unitary. (Hint - if it's been a while since you've played with trigonometric functions, you should look up angle addition formulas for sin and cos.)

4. Consider the single-qubit operation $I - 2|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is any valid qubit state and I is the 2×2 identity matrix. (This unitary will be important for the quantum searching algorithm)

   (a) Show $I - 2|\psi\rangle\langle\psi|$ is a unitary operation.

   (b) Describe what U does.

5. How long did you spend on this homework?