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Quantum Operations (gates)

- Reversible operations (unlike measurements)
- Represented by: unitary matrices.

$U^\dagger$ = conjugate transpose of U

U is unitary iff $UU^\dagger = U^\dagger U = I$

regular matrix multiplication

identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

matrix multiplication

If U acts on $|\psi\rangle$, state becomes $U|\psi\rangle = |\psi'\rangle$

Bra: $\langle\psi'| = \langle\psi|U^\dagger$

$U \Rightarrow 2\times 2$ matrix $\Rightarrow$ 1-qubit operation

$U \Rightarrow 4\times 4$ matrix $\Rightarrow$ 2-qubit operation

Questions

Q: If U is unitary and $|\psi\rangle$ is a state, is $U|\psi\rangle$ a state?

Q: Why is U reversible? (What type of matrix is $U^\dagger$?)

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A: Let $|\psi'\rangle = U|\psi\rangle$. $|\psi'\rangle$ is a state if it is properly normalized:

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|I|\psi\rangle$$
$$= \langle\psi|(I|\psi\rangle)$$
$$= \langle\psi|\psi\rangle = 1$$

A: If $|\psi\rangle \rightarrow U|\psi\rangle$, we can reverse U by applying $U^\dagger$ to get $U^\dagger U|\psi\rangle = |\psi\rangle$. But we can only apply $U^\dagger$ if $U^\dagger$ is unitary. Now $(U^\dagger)^\dagger \stackrel{?}{=} U$, so $U^\dagger (U^\dagger)^\dagger = U^\dagger U = I$, so $U^\dagger$ is unitary.

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- Famous Unitaries

Paulis: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hadamard: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ CNOT $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Other ways to represent unitaries:

Ket-bra

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 & 1\cdot 0 \\ 0\cdot 1 & 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

So $Z|+\rangle =$

$$\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)|+\rangle = |0\rangle\underbrace{\langle 0|+\rangle}_{\frac{1}{\sqrt{2}}} - |1\rangle\underbrace{\langle 1|+\rangle}_{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

$$\left(|\psi\rangle\langle\phi|\right)^{\dagger} = |\phi\rangle\langle\psi|$$

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Action on orthonormal basis:

ex: $|0\rangle \to |+\rangle$

$|1\rangle \to |-\rangle$

CNOT: $|00\rangle \to |00\rangle$

$|01\rangle \to |01\rangle$

$|10\rangle \to |11\rangle$

$|11\rangle \to |10\rangle$

Very useful when using ket notation

ex:

$$\text{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\text{CNOT}|00\rangle + \text{CNOT}|11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) = |+\rangle|0\rangle$$