S.KIMMEL

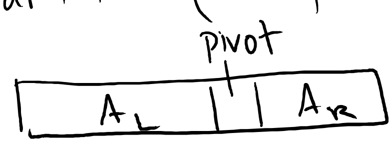
Goals

- Analyze expectation values using indicator random variables
- Analyze comparisons of 2 array elements

Input: Array A of length n, no repeated elements

Output: Array with sorted elements

QuickSort(array A)

1. If $|A| = 1$: return A
2. pivot = ChoosePivot(A)
3. Partition(A, pivot)
4. 

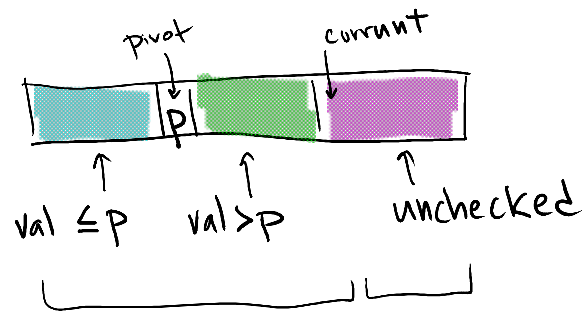
5. QuickSort(A_L)
6. QuickSort(A_R)

} Conquer

Partion

(1) Move pivot to front of array

(2) Maintain invariant



S. KIMMEL

Better Approach

1. Describe (generally) the sample space

   ex: $S = \{s : s$ is a sequence of possible pivot choices for our array$\}$

2. Describe (generally) the run time random variable

   ex: $R(s) =$ # of comparisons over algorithm with pivot choices s.

3. Write $R = \sum_i X_i$ ← other random variables

   ex: $X_{ij}(s) =$ # of times i^{th} largest element + j^{th} largest element are compared with pivot choices s

   $$R(s) = \sum_{i,j} X_{ij}(s)$$

   To create X_i, think about what are the basic events that are adding an amount of 1 to R

4. $$\mathbb{E}[R] = \sum_i \mathbb{E}[X_i]$$

   linearity of expectation

   $$\mathbb{E}[X_i] = \sum_{s \in S} X_i(s)\, p(s)$$

If choose X_i carefully, this is easier to calculate.

Best X_i = indicator random variable: takes value 0 or 1.

S. KIMMEL

Probability Example

Q: Suppose you have Q processors and J jobs. If you assign each job to a processor uniformly at random, what is the expected number of jobs processor 1 will get?

"Uniformly at random" = all equal probability

"expected" = average

A) $\sqrt{\frac{J}{Q}}$ B) $\sqrt{\frac{Q}{J}}$ $\boxed{\text{C) } \frac{J}{Q}}$ D) $\frac{Q}{J}$

We'll prove why!

Figuring Out Expectations:

1. Determine sample space.

Sample Space = $S = \{s : s \text{ is a possible outcome}\}$

S = Set of all possible ways J jobs can be assigned to Q processors: $\{1, 2, \ldots, Q\}^J$ = strings of length J with characters 1 to Q

processor $\to$ (4 10 1 ...)

job 1 job 2 job 3

S.KIMMEL

Q: What is $|S|$?

A) $J \cdot Q$ B) J^Q C) Q^J D) 2^{Q+J}

2. Define a <u>random variable</u> to represent quantity you care about. (Random variable = function $Y: S \to \mathbb{R}$)

$Y(s)$ = # of jobs assigned to processor 1 in assignment s.

Q: What is $Y(10, 4, 1, 3)$?

A) 0 B) 1 C) 3 D) 4

S.KIMMEL

Q: What is $|S|$?

A) $J \cdot Q$ B) J^Q C) Q^J D) 2^{Q+J}

Using product rule... Q^J

2. Define a <u>random variable</u> to represent quantity you care about. (Random variable = function $Y: S \to \mathbb{R}$)

$Y(s)$ = # of jobs assigned to processor 1 in assignment s.

Q: What is $Y(10, 4, 1, 3)$?

A) 0 B) 1 C) 3 D) 4

Only job 3 is assigned to processor 1

S. KIMMEL

3. Write main random variable as a sum of indicator variables.

Indicator variable is a random variable X_A where

$$X_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases} \quad \text{for } A \subseteq S$$

$$X_j(s) = \begin{cases} 1 & \text{if job } j \text{ assigned to Processor } 1 \text{ in } i \\ 0 & \text{else} \end{cases}$$

$$Y(s) = \sum_{j=1}^{J} X_j(s)$$

4. Use Linearity of Expectation

$\mathbb{E}[X]$ = expected value of variable X.

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$\Leftarrow$ Holds even if X, Y not independent random variables!

Our case: $\mathbb{E}[Y] = \sum_{j=1}^{J} \mathbb{E}[X_j]$

S. KIMMEL

5. Calculate Expected Value:

$$\mathbb{E}[Y] = \sum_{s \in S} \Pr[s] \cdot Y(s)$$

value of function X on input i

probability of outcome i

For indicator variable X_A

$$X_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

$$\mathbb{E}[X_E] = \sum_{s \in S} \Pr[s]\, X_A(s)$$

$$= \sum_{s \in E} \Pr(s) \cdot 1 + \sum_{s \notin A} \Pr(s) \cdot 0$$

$$= \sum_{s \in E} \Pr(s)$$

$$= \Pr[\text{event } A]$$

Definition of Probability of event A.

Want to choose indicator variables where figuring out this probability is easy

$$\mathbb{E}[Y] = \sum_{j=1}^{J} \mathbb{E}[X_j]$$

$$= \sum_{j=1}^{J} \Pr[j^{th} \text{ job assigned to } 1^{st} \text{ processor}]$$

$$= \sum_{j=1}^{J} \frac{1}{P} = \frac{J}{P}$$

S. KIMMEL

Back to QuickSort:

$R = \sum_{i,j} X_{ij}$ ← # comparisons b/t i^{th} largest + j^{th} largest array elements on element of sample space

Let z_i = i^{th} largest element of A

5	2	7
z_2	z_1	z_3

Story of z_i, z_j:

- As long as pivot $= z_k$ with $k > i, j$ or $k < i, j$, z_i, z_j get put together into same subarray for recursion. (No comparisons)
- Something interesting only happens when pivot is z_k with $i \leq k \leq j$

z_i	z_{i+1}	z_{i+2}	...	z_j

$k = i$ or j

z_i, z_j

1 comparison

(No further comparisons because pivot is not included in recursive calls)

$i < k < j$

z_i, z_j

0 comparisons

(No further comparisons because z_i and z_j separated: z_i in A_L, z_j in A_R.)

Because X_{ij} is 0 or 1, this means X_{ij} is an indicator random variable!